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# Dendrification of Hom-Malcev Algebras

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**Abstract.** The main goal of this work is to introduce a notion of Hom-M-dendriform algebras which can be seen as the dendriform version of Hom-Malcev algebras. In fact they are the algebraic structures behind the $\mathcal{O}$-operator of Hom-pre-Malcev algebras. They also fit into a larger framework as Hom-Malcev algebraic analogues of Hom-L-dendriform algebras. Furthermore, we show a connections between Hom-M-dendriform algebras and Hom-alternative quadri-algebras.

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**Keywords.** Hom-Malcev algebra, Hom-alternative algebra, Hom-M-dendriform algebra, Rota–Baxter operator, $\mathcal{O}$-operator.

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## 1. Introduction

Rota–Baxter operators were introduced by Baxter [7] in 1960 in the study of fluctuation theory in Probability. Such operators $\mathcal{R} : A \to A$ are defined on an algebra $A$ by the identity

$$\mathcal{R}(x)\mathcal{R}(y) = \mathcal{R}(\mathcal{R}(x)y + x\mathcal{R}(y) + \lambda xy), \tag{1.1}$$

for all $x, y \in A$, where $\lambda$ is a scalar called the weight of $\mathcal{R}$. These operators were further investigated by Rota [46], Atkinson [5], Cartier [10] and others. Since the late 1990 s, the study of Rota–Baxter operators has made great progress in many areas, both theoretically and in applications [2,17–22]. Rota–Baxter algebras allow us to define dendriform dialgebra ($\lambda = 0$, see [1] ) and trialgebra structures ($\lambda \neq 0$, see [16]) in associative algebras.

In this article, we deal with a specific type of algebraic structures, called Hom-algebraic structures. The defining identities of such algebraic structures are twisted using linear maps. These structures first appeared in the form of Hom-Lie algebras and more general quasi Hom-Lie algebras [25], in the context of the deformation of Witt and Virasoro algebras using general twisted derivations called $\sigma$-derivations. Hom-algebras have been studied extensively by Hartwig, Larsson, Makhlouf, Silvestrov, Yau and other authors [4,25,31–33,40,41,54]. Other types of algebras, such as associative, Leibniz, Poisson, Hopf, Malcev, etc., twisted by homomorphisms have also been studied in recent years.

Recently, there have been several interesting developments of Hom-Lie algebras in mathematics and mathematical physics, including Hom-Lie bialgebras [11,13], quadratic Hom-Lie algebras [8], involutive Hom-semigroups [56], deformed vector fields and differential calculus [30,33], representations [48], cohomology and homology theory [4,54], Hom-Yang–Baxter equations [12,48,52], Hom-Lie 2-algebras [49,50], $(m, n)$-Hom-Lie algebras [36], Hom-left-symmetric algebras [40] and enveloping algebras [53].

The notion of dendriform algebras was introduced in 1995 by Loday [34] in his study of algebraic $K$-theory. Dendriform algebras are algebras with two operations that dichotomise the notion of associative algebras. Later the notion of tridendriform algebra were introduced by Loday and Ronco in their study of polytopes and Koszul duality (see [35]). To determine the algebraic structures behind a pair of commuting Rota–Baxter operators (on an associative algebra), Aguiar and Loday introduced the notion of quadri-algebra in [3].

Another important class of nonassociative algebras is the Malcev algebras which were introduced in 1955 by Malcev [42], generalizing Lie algebras. They play an important role in the physics and geometry of smooth loops. Just as the tangent algebra of a Lie group is a Lie algebra, the tangent algebra of a locally analytic Moufang loop is a Malcev algebra [26,28,29,37,42,44,47], see also [24,43,45] for discussions about connections with physics.

The Hom-type generalization of Malcev algebras, called Hom-Malcev algebras, is defined in [55], where connections between Hom-alternative algebras and Hom-Malcev algebras are given.

**Definition 1.1.** [55] A Hom-Malcev algebra is a vector space $A$ together with a skew-symmetric bilinear map $[-,-] : A \otimes A \to A$ and a linear map $\alpha : A \to A$ such that, for all $x, y, z \in A$,

$$J_\alpha(\alpha(x), \alpha(y), [x, z]) = [J_\alpha(x, y, z), \alpha^2(x)], \tag{1.2}$$

where $J_\alpha(x, y, z) = [[x, y], \alpha(z)] + [[y, z], \alpha(x)] + [[z, x], \alpha(y)]$ is the Hom-Jacobian of $x, y, z$ ($\alpha$-Jacobian). The Hom-Malcev identity (1.2) is equivalent to

$$\begin{aligned}[\alpha([x, z]), \alpha([y, t])] = {} & [[[x, y], \alpha(z)], \alpha^2(t)] + [[[y, z], \alpha(t)], \alpha^2(x)] \\ & + [[[z, t], \alpha(x)], \alpha^2(y)] + [[[t, x], \alpha(y)], \alpha^2(z)].\end{aligned} \tag{1.3}$$

In [38], to find a dendriform algebra whose anti-commutator is a pre-Malcev algebra, M-dendriform algebras were introduced, an $\mathcal{O}$-operator (specifically a Rota–Baxter operator of weight zero) on a pre-Malcev algebra or two commuting Rota–Baxter operators on a Malcev algebra were shown to give an M-dendriform algebra, and the relationships between M-dendriform algebras and Loday algebras, especially quadri-algebras, were established.

Our main objective is to introduce the notion of Hom-M-dendriform algebras which are the twisted analogue of M-dendriform algebras, where the M-dendriform algebra identity is twisted by a self-linear map, called the structure map, and study, through Rota–Baxter operators and $\mathcal{O}$-operators, the relationship between Hom-pre-Malcev algebras and Hom-M-dendriform algebras.

Hom-M-dendriform algebra gives rise to a Hom-pre-Malcev algebra and a Hom-Malcev algebra, in the same way as Hom-L-dendriform algebra gives rise to a Hom-pre-Lie algebra and Hom-Lie algebra (see [14,15,39] for more details).

We study in the Sect. 2 the representation of Hom-Malcev algebras and recall the definitions and some properties of Hom-pre-Malcev algebras. The notion of a representation of a Hom-pre-Malcev algebra was introduced in [23]. However, some technical conditions must be added to obtain the dual representation, which is not convenient for applications. In this paper, we provide a natural formula for a dual representation which is nontrivial. This is presented in Sect. 3. In Sect. 4, we introduce the notion of Hom-M-dendriform algebra and then study some of their fundamental properties in terms of the $\mathcal{O}$-operators of Hom-pre-Malcev algebras. Then we define a Hessian structure on a regular Hom-pre-Malcev algebra $(A, \cdot, \alpha)$ to be a nondegenerate 2-cocycle $\mathcal{B} \in Sym^2 A^*$, and show that there is a relation between Hom-M-dendriform algebras and Hessian structures on $(A, \cdot, \alpha)$. Their relationship with Hom-Malcev algebras and Hom-alternative quadri-algebras are also described in Sect. 5.

## 2. Basics on Hom-Malcev and Hom-Pre-Malcev Algebras

In this section, we recall representations and dual representations of Hom-Malcev algebras and the definitions of Hom-pre-Malcev algebras which are useful throughout this paper.

In this paper, all vector spaces are over a field $\mathbb{K}$ of characteristic 0.

A Hom-Malcev algebra $(A, [-,-], \alpha)$ is said to be a **regular Hom-Malcev algebra** if $\alpha$ is invertible. We denote by $End(A)$ the space of all linear maps $f : A \to A$.

The notion of representation of Hom-Malcev algebra was introduced in [23,28].

**Definition 2.1.** Let $(A, [-,-], \alpha)$ be a Hom-Malcev algebra. A representation of $(A, [-,-], \alpha)$ on a vector space $V$ with respect to $\beta \in End(V)$ is a linear map $\rho : A \longrightarrow End(V)$ satisfying, for any $x, y, z \in A$,

$$\rho(\alpha(x))\beta = \beta\rho(x), \tag{2.1}$$

$$\begin{aligned}\rho([[x,y],\alpha(z)])\beta^2 = {}& \rho(\alpha^2(x))\rho(\alpha(y))\rho(z) - \rho(\alpha^2(z))\rho(\alpha(x))\rho(y)\\ & +\rho(\alpha^2(y))\rho([z,x])\beta - \rho(\alpha([y,z]))\rho(\alpha(x))\beta.\end{aligned} \tag{2.2}$$

We denote the representation by $(V, \rho, \beta)$.

**Example 2.1.** Let $(A, [-,-], \alpha)$ be a Hom-Malcev algebra. For any integer $s$, the $\alpha^s$-adjoint representation of $(A, [-,-], \alpha)$ on $A$, which we denote by $ad^s$, is defined for all $x, y \in A$ by

$$ad^s_x(y) = [\alpha^s(x), y].$$

In particular, we write $ad$ for $ad^0$.

**Example 2.2.** Let $(V, \rho, \beta)$ be a representation of a regular Hom-Malcev algebra $(A, [-,-], \alpha)$ and let $V^*$ be the dual vector space of $V$ with $\beta$ invertible. Let us define $\rho^\star : A \longrightarrow End(V^*)$ by

$$\rho^\star(x)(\xi) = \rho^*(\alpha(x))\big((\beta^{-2})^*(\xi)\big) = -\langle \xi, \rho(\alpha^{-1}(x))(\beta^{-2}(a))\rangle, \quad \forall x \in A, \xi \in V^*, \tag{2.3}$$

where $\langle -,- \rangle$ denote the canonical pairing between the dual space $V^*$ and the vector space $V$. Then $\rho^\star : A \longrightarrow End(V^*)$ is a representation of $(A, [-,-], \alpha)$ on $V^*$ with respect to $(\beta^{-1})^*$, which is called the dual representation of $(V, \rho, \beta)$.

**Proposition 2.1.** *Let $(A, [-,-], \alpha)$ be a Hom-Malcev algebra, $(V, \beta)$ a vector space, and $\rho : A \longrightarrow End(V)$ a linear map. Then $(V, \rho, \beta)$ is a representation of $A$ if and only if $(A \oplus V, [-,-]_\rho, \alpha + \beta)$ is a Hom-Malcev algebra, where $[-,-]_\rho$ and $\alpha + \beta$ are defined for all $x, y \in A$, $a, b \in V$ by*

$$\begin{aligned}[x + a, y + b]_\rho &= [x, y] + \rho(x)a - \rho(y)b,\\ (\alpha + \beta)(x + a) &= \alpha(x) + \beta(a).\end{aligned}$$

*This Hom-Malcev algebra is called the semi-direct product of $(A, [-,-], \alpha)$ and $(V, \beta)$, and denoted by $A \ltimes_\rho^{\alpha,\beta} V$ or simply $A \ltimes V$.*

The notion of $\mathcal{O}$-operator is useful tool for constructing solutions of the classical Yang–Baxter equation. Let $(A, [-,-], \alpha)$ be a Hom-Malcev algebra and $(V, \rho, \beta)$ a representation. A linear map $T : V \to A$ is called an $\mathcal{O}$-operator associated to $(V, \rho, \beta)$ if for all $a, b \in V$,

$$\alpha \circ T = T \circ \beta, \qquad [T(a), T(b)] = T\big(\rho(T(a))b - \rho(T(b))a\big). \tag{2.4}$$

**Example 2.3.** A Rota–Baxter operator of weight 0 on a Hom-Malcev algebra $A$ is just an $\mathcal{O}$-operator associated to the adjoint representation $(A, ad, \alpha)$, that is, $\mathcal{R}$ satisfies for all $x, y \in A$,

$$\mathcal{R} \circ \alpha = \alpha \circ \mathcal{R}, \qquad [\mathcal{R}(x), \mathcal{R}(y)] = \mathcal{R}([\mathcal{R}(x), y] + [x, \mathcal{R}(y)]).$$

Now, we recall the definition of Hom-pre-Malcev algebra and exhibit construction results in terms of $\mathcal{O}$-operators on Hom-Malcev algebras (for more details, see [23]).

**Definition 2.2.** Let $A$ be a vector space with a bilinear product " $\cdot$" $A \times A \to A$ and $\alpha : A \to A$ is a linear map. The triple $(A, \cdot, \alpha)$ is called a **Hom-pre-Malcev algebra** if for all $x, y, z, t \in A$,

$$\begin{aligned} &\alpha([y,z]) \cdot \alpha(x \cdot t) + [[x,y], \alpha(z)] \cdot \alpha^2(t) \\ &\quad + \alpha^2(y) \cdot ([x,z] \cdot \alpha(t)) - \alpha^2(x) \cdot (\alpha(y) \cdot (z \cdot t)) + \alpha^2(z) \cdot (\alpha(x) \cdot (y \cdot t)) = 0, \end{aligned} \tag{2.5}$$

where $[x, y] = x \cdot y - y \cdot x$.

The identity (2.5) is equivalent to $HPM(x, y, z, t) = 0$, where for all $x, y, z, t \in A$,

$$\begin{aligned} 0 = {} & \alpha(y \cdot z) \cdot \alpha(x \cdot t) - \alpha(z \cdot y) \cdot \alpha(x \cdot t) \\ & + ((x \cdot y) \cdot \alpha(z)) \cdot \alpha^2(t) - ((y \cdot x) \cdot \alpha(z)) \cdot \alpha^2(t) \\ & - (\alpha(z) \cdot (x \cdot y)) \cdot \alpha^2(t) + (\alpha(z) \cdot (y \cdot x)) \cdot \alpha^2(t) \\ & + \alpha^2(y) \cdot ((x \cdot z) \cdot \alpha(t)) - \alpha^2(y) \cdot ((z \cdot x) \cdot \alpha(t)) \\ & - \alpha^2(x) \cdot (\alpha(y) \cdot (z \cdot t)) + \alpha^2(z) \cdot (\alpha(x) \cdot (y \cdot t)). \end{aligned} \tag{2.6}$$

A Hom-pre-Malcev algebra is said to be a **regular Hom-pre-Malcev algebra** if $\alpha$ is invertible.

**Definition 2.3.** Let $(A_1, \cdot_1, \alpha_1)$ and $(A_2, \cdot_2, \alpha_2)$ be two Hom-pre-Malcev algebras. A linear map $f : A_1 \to A_2$ is called a morphism of Hom-pre-Malcev algebras if, for all $x, y \in A$,

$$f(x) \cdot_2 f(y) = f(x \cdot_1 y), \qquad f \circ \alpha_1 = \alpha_2 \circ f.$$

**Proposition 2.2.** *Let $(A, \cdot, \alpha)$ be a Hom-pre-Malcev algebra. Then, the commutator $[x, y] = x \cdot y - y \cdot x$ defines a Hom-Malcev algebra on $A$. It is called the* ***sub-adjacent Hom-Malcev algebra*** *of $(A, \cdot, \alpha)$ and $(A, \cdot, \alpha)$ is called the* ***compatible Hom-pre-Malcev algebra*** *on $(A, [-,-], \alpha)$. The subadjacent Hom-Malcev algebra $(A, \cdot, \alpha)$ is denoted by $A^C$.*

**Proposition 2.3.** *Let $(A, [-,-], \alpha)$ be a Hom-Malcev algebra and $T : V \to A$ be an $\mathcal{O}$-operator of $A$ associated with a representation $(V, \rho, \beta)$. Then $(V, \cdot, \beta)$ is a Hom-pre-Malcev algebra with the product "$\cdot$" defined for all $a, b \in V$ by*

$$a \cdot b = \rho(T(a))b. \tag{2.7}$$

An obvious consequence of Proposition 2.3 is the construction of a Hom-pre-Malcev algebra in terms of a Rota–Baxter operator of weight zero of a Hom-Malcev algebra.

**Corollary 2.1.** *Let $\mathcal{R} : A \longrightarrow A$ is a Rota–Baxter operator on a Hom-Malcev algebra $(A, [-,-], \alpha)$. Then $(A, \cdot, \alpha)$ is a Hom-pre-Malcev algebra, where for all $x, y \in A$,*

$$x \cdot y = [\mathcal{R}(x), y].$$

## 3. Representation of Hom-Pre-Malcev Algebras

Now, we introduce the definition of the representation of Hom-pre-Malcev algebras and provide some related results. We construct the dual representation of a representation of a Hom-pre-Malcev algebra. In addition, we introduce the notion of $\mathcal{O}$-operators of a Hom-pre-Malcev algebra which is a generalisation of Rota–Baxter operators.

**Definition 3.1.** A representation of Hom-pre-Malcev algebra $(A, \cdot, \alpha)$ on a vector space $V$ consists of a triple $(\ell, r, \beta)$, where $\ell : A \to End(V)$ is a representation of the sub-adjacent Hom-Malcev algebra $A^C$ on $V$ with respect to $\beta \in End(V)$ and $r : A \to End(V)$ is a linear map such that for all $x, y, z \in A$,

$$\beta r(x) = r(\alpha(x))\beta, \tag{3.1}$$

$$\begin{aligned} &r(\alpha^2(x))\rho(\alpha(y))\rho(z) - r(\alpha(z)\cdot(y\cdot x))\beta^2 + \ell(\alpha^2(y))r(z\cdot x)\beta \\ &\quad +\ell(\alpha([y,z]))r(\alpha(x))\beta - \ell(\alpha^2(z))r(\alpha(x))\rho(y) = 0, \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\ell(\alpha^2(y))\ell(\alpha(z))r(x) - r(\alpha^2(x))\rho(\alpha(y))\rho(z) - \ell(\alpha^2(z))r(y\cdot x)\beta \\ &\quad -r(\alpha(z\cdot x))\rho(\alpha(y))\beta + r([z,y]\cdot\alpha(x))\beta^2 = 0, \end{aligned} \tag{3.3}$$

$$\begin{aligned} &r(\alpha(y)\cdot(z\cdot x))\beta^2 + r(\alpha^2(x))\rho([y,z])\beta - \ell(\alpha^2(y))\ell(\alpha(z))r(x) \\ &\quad +r(\alpha(y\cdot x))\rho(\alpha(z))\beta + \ell(\alpha^2(z))r(\alpha(x))\rho(y) = 0, \end{aligned} \tag{3.4}$$

where $\rho(x) = \ell(x) - r(x)$ and $[x, y] = x \cdot y - y \cdot x$.

We denote the representation of a Hom-pre-Malcev algebra $(A, \cdot, \alpha)$ by $(V, \ell, r, \beta)$. Now, define a linear operation $\cdot_\ltimes : \otimes^2(A \oplus V) \longrightarrow (A \oplus V)$ by

$$(x + a) \cdot_\ltimes (y + b) = x \cdot y + \ell(x)(b) + r(y)(a),\ \forall x, y \in A, a, b \in V,$$

and a linear map $\alpha + \beta : A \oplus V \longrightarrow A \oplus V$ by

$$(\alpha + \beta)(x + a) = \alpha(x) + \beta(a),\ \forall x \in A, a \in V.$$

**Proposition 3.1.** [23] *The triple $(A \oplus V, \cdot_\ltimes, \alpha + \beta)$ is a Hom-pre-Malcev algebra.*

The Hom-pre-Malcev algebra $(A \oplus V, \cdot_\ltimes, \alpha + \beta)$ is denoted by $A \ltimes^{\alpha}_{(\ell,\ r)} V$, or simply $A \ltimes V$, and called the semi-direct product of the Hom-pre-Malcev algebra $(A, \cdot, \alpha)$ and representation $(V, \ell, r, \beta)$.

**Example 3.1.** Let $(A, \cdot, \alpha)$ be a Hom-pre-Malcev algebra. Define the left multiplication $L : A \to End(A)$ by $L_x(y) = x \cdot y$ for all $x, y \in A$. Then $(A, L, \alpha)$ is a representation of the Hom-Malcev algebra $A^c$. Moreover, we define the right multiplication $R : A \to End(A)$ is defined by $R_x(y) = y \cdot x$. It is obvious

that $(A, L, R, \alpha)$ is a representation of a Hom-pre-Malcev algebra on itself, which is called **the regular representation**.

**Proposition 3.2.** *Let $(A, \cdot, \alpha)$ be a multiplicative Hom-pre-Malcev algebra. For any integer $s$, define $L^s, R^s : A \to End(A)$ for all $x, y \in A$ by*

$$L^s_x(y) = \alpha^s(x) \cdot y, \quad R^s_x(y) = y \cdot \alpha^s(x).$$

*Then, $(A, L^s, R^s, \alpha)$ is a representation of a Hom-pre-Malcev algebra $(A, \cdot, \alpha)$. Note that $L^0 = L$ and $R^0 = R$, where $(L, R)$ is the regular representation with respect $\alpha$ defined in Example 3.1.*

*Proof.* For all $x, y, z, t \in A$, we have

$$L^s_{\alpha(x)}\alpha(y) = \alpha^{s+1}(x) \cdot \alpha(y) = \alpha(\alpha^s(x) \cdot y) = \alpha(L^s_x y).$$

This implies that $L^s_{\alpha(x)} \circ \alpha = \alpha \circ L^s_x$. Similarly, we have $R^s_{\alpha(x)} \circ \alpha = \alpha \circ R^s_x$.

By the definition of a Hom-pre-Malcev algebra,

$$\begin{aligned}
&L^s_{\alpha[y,z]}L^s_{\alpha(x)}\alpha(t) + L^s_{[[x,y],\alpha(z)]}\alpha^2(t)\\
&\quad = \alpha^{s+1}([y,z]) \cdot (\alpha^{s+1}(x) \cdot \alpha(t)) + \alpha^s([[x,y],\alpha(z)]) \cdot \alpha^2(t)\\
&\quad = \alpha^{s+2}(y) \cdot (\alpha^s([z,x]) \cdot \alpha(t)) + \alpha^{s+2}(x) \cdot (\alpha^{s+1}(y) \cdot (\alpha^s(z) \cdot t))\\
&\qquad - \alpha^{s+2}(z) \cdot (\alpha^{s+1}(x) \cdot (\alpha^s(y) \cdot t))\\
&\quad = L^s_{\alpha^2(y)}L^s_{[z,x]}\alpha(t) + L^s_{\alpha^2(x)}L^s_{\alpha(y)}L^s_z(t) - L^s_{\alpha^2(z)}L^s_{\alpha(x)}L^s_x(t) = 0.
\end{aligned}$$

Then $(A, L^s, \alpha)$ is a representation of the sub-adjacent Hom-Malcev algebra $A^C$. Furthermore,

$$\begin{aligned}
&R^s_{\alpha^2(x)}ad^s_{\alpha(y)}ad^s_z(t) + L^s_{\alpha([y,z])}R^s_{\alpha(x)}\alpha(t)\\
&\quad = [\alpha^{s+1}(y), [\alpha^s(z), t]] \cdot \alpha^{s+2}(x) + \alpha^{s+1}([y,z]) \cdot (\alpha(t) \cdot \alpha^s(x))\\
&\quad = \alpha^{s+2}(z) \cdot ([\alpha^s(y), t] \cdot \alpha^{s+1}(x)) + \alpha^2(t) \cdot (\alpha^{s+1}(z) \cdot \alpha^s(y \cdot x))\\
&\qquad - \alpha^{s+2}(y) \cdot (\alpha(t) \cdot \alpha^s(z \cdot x))\\
&\quad = L^s_{\alpha^2(z)}R^s_{\alpha(x)}ad^s_y(t) + R^s_{\alpha(z)\cdot(y\cdot x)}\alpha^2(t) + L^s_{\alpha^2(y)}R^s_{z\cdot x}\alpha(t).
\end{aligned}$$

This implies that (3.2) holds. Similarly, (3.3) and (3.4) hold, too.

□

**Proposition 3.3.** *Let $(V, \ell, r, \beta)$ be a representation of a Hom-pre-Malcev algebra $(A, \cdot, \alpha)$. Then, $(V, \ell - r, \beta)$ is a representation of $(A, [-,-], \alpha)$.*

Let $(V, \ell, r, \beta)$ be a representation of a Hom-pre-Malcev algebra $(A, \cdot, \alpha)$. In the sequel, we always assume that $\beta$ is invertible. For all $x \in A$, $a \in V$, $\xi \in V^*$ define $\ell^*, r^* : A \longrightarrow End(V^*)$ as usual by

$$\langle \ell^*(x)(\xi), a\rangle = -\langle \xi, \ell(x)(a)\rangle, \quad \langle r^*(x)(\xi), a\rangle = -\langle \xi, r(x)(a)\rangle.$$

Then define $\ell^\star, r^\star : A \longrightarrow End(V^*)$ by

$$\ell^\star(x)(\xi) = \ell^*(\alpha(x))\big((\beta^{-2})^*(\xi)\big), \tag{3.5}$$

$$r^\star(x)(\xi) = r^*(\alpha(x))\big((\beta^{-2})^*(\xi)\big). \tag{3.6}$$

**Theorem 3.1.** *Let $(V, \ell, r, \beta)$ be a representation of a Hom-pre-Malcev algebra $(A, \cdot, \alpha)$. The quadruple $(V^*, \ell^\star - r^\star, -r^\star, (\beta^{-1})^*)$ is a representation of $(A, \cdot, \alpha)$. It is called the* ***dual representation*** *of $(V, \ell, r, \beta)$.*

*Proof.* Since $(V, \ell, r, \beta)$ is a representation of a Hom-pre-Malcev algebra $(A, \cdot, \alpha)$, by Proposition 3.3, we deduce that $(V, \ell - r, \beta)$ is a representation of the sub-adjacent $A^C$. Moreover, by Lemma 2.2, $(V, (\ell - r)^\star = \ell^\star - r^\star, (\beta^{-1})^*)$ is a representation of the Hom-Malcev algebra $A^C$. By (3.1) and (3.6), for all $x \in A,\ \xi \in V^*$,

$$\begin{aligned}-r^\star(\alpha(x))((\beta^{-1})^*(\xi)) &= -r^*(\alpha^2(x))(\beta^{-3})^*(\xi) = (\beta^{-1})^*(-r^*(\alpha(x))(\beta^{-2})^*(\xi))\\ &= (\beta^{-1})^*(-r^\star(x)(\xi)),\end{aligned}$$

which yields $-r^\star\big(\alpha(x)\big) \circ (\beta^{-1})^* = (\beta^{-1})^* \circ (-r^\star(x))$. We check only that $(V^*, \ell^\star - r^\star, -r^\star, (\beta^{-1})^*)$ verifies (3.2). Let $x, y, z \in A, \xi \in V^*,\ a \in V$ and according to (2.1), (3.1), (3.2) and (3.6), we have

$$\begin{aligned}
&\Big\langle r^\star(\alpha(z)\cdot(y\cdot x))((\beta^{-2})^*(\xi)), a\Big\rangle = \Big\langle r^*(\alpha^2(z)\cdot(\alpha(y)\cdot\alpha(x)))((\beta^{-4})^*(\xi)), a\Big\rangle\\
&= -\Big\langle((\beta^{-4})^*(\xi)), r(\alpha^2(z)\cdot(\alpha(y)\cdot\alpha(x)))(a)\Big\rangle\\
&= -\Big\langle((\beta^{-4})^*(\xi)), r(\alpha^3(x))(\rho(\alpha^2(y))(\rho(\alpha(z))(\beta^{-2}(a)))) + \ell(\alpha^3(y))(r(\alpha(z)\cdot\alpha(x))(\beta^{-1}(a)))\\
&\quad + \ell(\alpha^2([y,z]))(r(\alpha^2(x))(\beta^{-1}(a))) - \ell(\alpha^3(z))(r(\alpha^2(x))(\rho(\alpha(y))(\beta^{-2}(a))))\Big\rangle\\
&= -\Big\langle((\beta^{-6})^*(\xi)), r(\alpha^5(x))(\rho(\alpha^4(y))(\rho(\alpha^3(z)))) + \ell(\alpha^5(y))(r(\alpha^3(z)\cdot\alpha^3(x))(\beta(a)))\\
&\quad + \ell(\alpha^4([y,z]))(r(\alpha^4(x))(\beta(a))) - \ell(\alpha^5(z))(r(\alpha^4(x))(\rho(\alpha^3(y))))\Big\rangle\\
&= -\Big\langle - \rho^*(\alpha^3(z))(\rho^*(\alpha^4(y))(r^*(\alpha^5(x))((\beta^{-6})^*(\xi))))\\
&\quad + r^*(\alpha^2(z)\cdot\alpha^2(x))(\ell^*(\alpha^4(y))((\beta^{-5})^*(\xi)))\\
&\quad + r^*(\alpha^3(x))(\ell^*(\alpha^3([y,z]))((\beta^{-5})^*(\xi))) + \rho^*(\alpha^3(y))(r^*(\alpha^4(x))(\ell^*(\alpha^5(z))((\beta^{-6})^*(\xi)))), a\Big\rangle\\
&= -\Big\langle - \rho^*(\alpha^3(z))(\rho^*(\alpha^4(y))(r^\star(\alpha^4(x))((\beta^{-4})^*(\xi))))\\
&\quad + r^*(\alpha^2(z)\cdot\alpha^2(x))(\ell^\star(\alpha^3(y))((\beta^{-3})^*(\xi)))\\
&\quad + r^*(\alpha^3(x))(\ell^\star(\alpha^2([y,z]))((\beta^{-3})^*(\xi))) + \rho^*(\alpha^3(y))(r^*(\alpha^4(x))(\ell^\star(\alpha^4(z))((\beta^{-4})^*(\xi)))), a\Big\rangle\\
&= -\Big\langle - \rho^*(\alpha^3(z))(\rho^*(\alpha^4(y))((\beta^{-4})^*(r^\star(x)(\xi)))) + r^*(\alpha^2(z)\cdot\alpha^2(x))((\beta^{-3})^*(\ell^\star(y)(\xi)))\\
&\quad + r^*(\alpha^3(x))((\beta^{-2})^*(\ell^\star([y,z])((\beta^{-1})^*(\xi)))) + \rho^*(\alpha^3(y))(r^*(\alpha^4(x))((\beta^{-4})^*(\ell^\star(z)(\xi)))), a\Big\rangle\\
&= -\langle-\rho^*(\alpha^3(z))(\rho^\star(\alpha^3(y))((\beta^{-2})^*(r^\star(x)(\xi)))) + r^\star(\alpha(z)\cdot\alpha(x))((\beta^{-1})^*(\ell^\star(y)(\xi)))\\
&\quad + r^\star(\alpha^2(x))(\ell^\star([y,z])((\beta^{-1})^*(\xi))) + \rho^*(\alpha^3(y))(r^*(\alpha^3(x))((\beta^{-2})^*(\ell^\star(z)(\xi)))), a\Big\rangle\\
&= -\langle-\rho^*(\alpha^3(z))((\beta^{-2})^*(\rho^\star(\alpha(y))(r^\star(x)(\xi)))) + r^\star(\alpha(z)\cdot\alpha(x))(\ell^\star(\alpha(y))((\beta^{-1})^*(\xi)))\\
&\quad + r^\star(\alpha^2(x))(\ell^\star([y,z])((\beta^{-1})^*(\xi))) + \rho^*(\alpha^3(y))((\beta^{-2})^*(r^\star(\alpha(x))(\ell^\star(z)(\xi)))), a\Big\rangle\\
&= -\langle-\rho^\star(\alpha^2(z))(\rho^\star(\alpha(y))(r^\star(x)(\xi))) + r^\star(\alpha(z)\cdot\alpha(x))(\ell^\star(\alpha(y))((\beta^{-1})^*(\xi)))\\
&\quad + r^\star(\alpha^2(x))(\ell^\star([y,z])((\beta^{-1})^*(\xi))) + \rho^*(\alpha^2(y))(r^\star(\alpha(x))(\ell^\star(z)(\xi))), a\Big\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}&r^\star(\alpha(z)\cdot(y\cdot x))(\beta^{-2})^* - \rho^\star(\alpha^2(z))\rho^\star(\alpha(y))r^\star(x) + r^\star(\alpha(z)\cdot\alpha(x))(\ell^\star - r^\star)(\alpha(y))(\beta^{-1})^*\\ &\quad + r^\star(\alpha^2(x))(\ell^\star - r^\star)([y,z])(\beta^{-1})^* + \rho^\star(\alpha^2(y))r^\star(\alpha(x))(\ell^\star - r^\star)(z) = 0.\end{aligned}$$

The other identities can be verified in a similar manner. Therefore, $(V^*, \ell^\star - r^\star, -r^\star, (\beta^{-1})^*)$ represents $(A, \cdot, \alpha)$.

□

Theorem 3.1 yields the following result for the dual representation of the regular representation of a regular Hom-pre-Malcev algebra.

**Corollary 3.1.** *Let $(A, \cdot, \alpha)$ be a regular Hom-pre-Malcev algebra, and let $A^*$ be the dual of the vector space $A$. Then $(A^*, ad^\star = L^\star - R^\star, -R^\star, (\alpha^{-1})^*)$ is a representation of $(A, \cdot, \alpha)$.*

**Proposition 3.4.** *Let $(V, \ell, r, \beta)$ be a representation of a Hom-pre-Malcev algebra $(A, \cdot, \alpha)$, where $V$ is finite-dimensional and $\beta$ is invertible. Then, the dual representation of $(V^*, \ell^\star - r^\star, -r^\star, (\beta^{-1})^*)$ is $(V, \rho, r, \beta)$.*

*Proof.* Clearly, $(V^*)^* \simeq V$ and $(((\beta^{-1})^*)^{-1})^* = \beta$. By [9, Lemma 3.7], we obtain

$$(\ell^\star - r^\star)^\star = (\rho^\star)^\star = \rho.$$

Similarly, we have $(-(-r^\star))^\star = r$. Then, the dual representation of $(V^*, \ell^\star - r^\star, -r^\star, (\beta^{-1})^*)$ is $(V, \rho, r, \beta)$.

□

As in [6,27], we rephrase the definition of $\mathcal{O}$-operator in terms of Hom-pre-Malcev algebras.

**Definition 3.2.** Let $(A, \cdot, \alpha)$ be a Hom-pre-Malcev algebra and $(V, \ell, r, \beta)$ be a representation. A linear map $T : V \to A$ is called an $\mathcal{O}$-operator associated to $(V, \ell, r, \beta)$ if $T$ satisfies

$$T \circ \beta = \alpha \circ T, \tag{3.7}$$

$$T(a) \cdot T(b) = T\big(\ell(T(a))b + r(T(b))a\big), \quad \forall a, b \in V. \tag{3.8}$$

**Example 3.2.** Let $T : V \to A$ be an $\mathcal{O}$-operator on a pre-Malcev algebra $(A, \cdot)$ with respect to the representation $(V, \ell, r)$. A pair $(\phi_A, \phi_V)$ is an endomorphism of the $\mathcal{O}$-operator $T$ if

$$\begin{aligned} T \circ \phi_V &= \phi_A \circ T, \\ \ell(\phi_A(x))(\phi_V(b)) &= \phi_V(\ell(x)(b)) \quad \text{and} \\ r(\phi_A(x))(\phi_V(b)) &= \phi_V(r(x)(b)), \quad \text{for all } x \in A,\ b \in V. \end{aligned}$$

Let us consider the Hom-pre-Malcev algebra $(A, \cdot_{\phi_A}, \phi_A)$ obtained by composition, where the Hom-pre-Malcev product is given by

$$x \cdot_{\phi_A} y := \phi_A(x \cdot y).$$

If we consider the compositions $\ell_{\phi_V} := \phi_V \circ \ell$ and $r_{\phi_V} := \phi_V \circ r$, then the triple $(V, \ell_{\phi_V}, r_{\phi_V}, \phi_V)$ is a representation of $(A, \cdot_{\phi_A}, \phi_A)$. Moreover, for all $a, b \in V$,

$$\begin{aligned} T(a) \cdot_{\phi_A} T(b) &= \phi_A(T(a) \cdot T(b)) = \phi_A\big(T(\ell(T(a))b + r(T(b))a)\big), \\ T\big(\ell_{\phi_V}(T(a))b + r_{\phi_V}(T(b))a\big) &= T\big(\phi_V(\ell(T(a))b + r(T(b))a)\big). \end{aligned}$$

It follows that the map $T : V \to A$ is an $\mathcal{O}$-operator on the Hom-pre-Malcev algebra $(A, \cdot_{\phi_A}, \phi_A)$ with respect to the representation $(V, \ell_{\phi_V}, r_{\phi_V}, \phi_V)$.

## 4. Hom-M-Dendriform Algebras

The goal of this section is to introduce the notion of Hom-M-dendriform algebras which is the dendriform version of Hom-Malcev algebras, and study the relationships with Hom-pre-Malcev algebras and Hom-Malcev algebras in terms of the $\mathcal{O}$-operators of Hom-pre-Malcev algebras.

### 4.1. Definitions and Some Basic Properties

**Definition 4.1.** A **Hom-M-dendriform algebra** is a vector space $A$ endowed with two bilinear products $\blacktriangleright, \blacktriangleleft$: $A \times A \to A$ and a linear map $\alpha : A \to A$ such that for all $x, y, z, t \in A$ and

$$x \cdot y = x \blacktriangleleft y + x \blacktriangleright y, \tag{4.1}$$

$$x \diamond y = x \blacktriangleleft y - y \blacktriangleright x, \tag{4.2}$$

$$[x, y] = x \cdot y - y \cdot x = x \diamond y - y \diamond x, \tag{4.3}$$

the following identities are satisfied:

$$\begin{aligned}&(\alpha(z) \diamond (y \diamond x)) \blacktriangleright \alpha^2(t) - \alpha^2(x) \blacktriangleright (\alpha(y) \cdot (z \cdot t)) + \alpha^2(z) \blacktriangleleft (\alpha(x) \blacktriangleright (y \cdot t))\\ &\quad +\alpha([y, z]) \blacktriangleleft \alpha(x \blacktriangleright t) - \alpha^2(y) \blacktriangleleft ((z \diamond x) \blacktriangleright \alpha(t)) = 0,\end{aligned} \tag{4.4}$$

$$\begin{aligned}&\alpha^2(z) \blacktriangleleft (\alpha(x) \blacktriangleleft (y \blacktriangleright t)) - (\alpha(z) \diamond (x \diamond y)) \blacktriangleright \alpha^2(t) - \alpha^2(x) \blacktriangleleft (\alpha(y) \blacktriangleright (z \cdot t))\\ &\quad -\alpha(z \diamond y) \blacktriangleright \alpha(x \cdot t) + \alpha^2(y) \blacktriangleright ([x, z] \cdot \alpha(t)) = 0,\end{aligned} \tag{4.5}$$

$$\begin{aligned}&\alpha^2(z) \blacktriangleleft (\alpha(x) \cdot (y \cdot t)) + ([x, y] \diamond \alpha(z)) \blacktriangleright \alpha^2(t) - \alpha^2(x) \blacktriangleleft (\alpha(y) \blacktriangleleft (z \blacktriangleright t))\\ &\quad +\alpha(y \diamond z) \blacktriangleright \alpha(x \cdot t) + \alpha^2(y) \blacktriangleleft ((x \diamond z) \blacktriangleright \alpha(t)) = 0,\end{aligned} \tag{4.6}$$

$$\begin{aligned}&[[x, y], \alpha(z)] \blacktriangleleft \alpha^2(t) - \alpha^2(x) \blacktriangleleft (\alpha(y) \blacktriangleleft (z \blacktriangleleft t)) + \alpha^2(z) \blacktriangleleft (\alpha(x) \blacktriangleleft (y \blacktriangleleft t))\\ &\quad +\alpha([y, z]) \blacktriangleleft \alpha(x \blacktriangleleft t) + \alpha^2(y) \blacktriangleleft ([x, z] \blacktriangleleft \alpha(t)) = 0.\end{aligned} \tag{4.7}$$

Otherwise, a Hom-M-dendriform algebra $(A, \blacktriangleright, \blacktriangleleft, \alpha)$ is said to be **regular** if $\alpha$ is invertible, and **multiplicative** if $\alpha(x \blacktriangleright y) = \alpha(x) \blacktriangleright \alpha(y)$ and $\alpha(x \blacktriangleleft y) = \alpha(x) \blacktriangleleft \alpha(y)$.

Let $(A_1, \blacktriangleright_1, \blacktriangleleft_1, \alpha_1))$ and $(A_2, \blacktriangleright_2, \blacktriangleleft_2, \alpha_2)$ be two Hom-M-dendriform algebras. A weak morphism $f : A_1 \longrightarrow A_2$ is a linear map such that

$$f \blacktriangleright_1 = \blacktriangleright_2 f^{\otimes 2}, \qquad f \blacktriangleleft_1 = \blacktriangleleft_2 f^{\otimes 2}.$$

A morphism $f : A_1 \longrightarrow A_2$ is a weak morphism such that $f\alpha_1 = \alpha_2 f$.

Now, we provide a way to construct a Hom-M-dendriform algebras starting from an M-dendriform algebra and weak self-morphism.

**Proposition 4.1.** *Let $(A, \blacktriangleright, \blacktriangleleft)$ be an M-dendriform algebra and $\alpha : A \to A$ be a weak morphism. Then $A_\alpha = (A, \blacktriangleright_\alpha = \alpha \blacktriangleright,\ \blacktriangleleft_\alpha = \alpha \blacktriangleleft,\ \alpha)$ is a Hom-M-dendriform algebra.*

*Moreover, if $\alpha$ is a morphism, then $A_\alpha$ is a multiplicative Hom-M-dendriform algebra.*

*Proof.* We only prove that $(A, \blacktriangleright_\alpha, \blacktriangleleft_\alpha, \alpha)$ satisfies the first Hom-M-dendriform identity. The other identities for $A_\alpha$ being a Hom-M-dendriform algebra can be verified similarly. In fact, for any $x, y, z, t \in A$, we know that

$$(z \diamond (y \diamond x)) \blacktriangleright t - x \blacktriangleright (y \cdot (z \cdot t)) + z \blacktriangleleft (x \blacktriangleright (y \cdot t)) + [y, z] \blacktriangleleft (x \blacktriangleright t) - y \blacktriangleleft ((z \diamond x) \blacktriangleright t) = 0.$$

Now applying $\alpha^3$ to the previous identity, we obtain

$$(\alpha^3(z) \diamond (\alpha^3(y) \diamond \alpha^3(x))) \blacktriangleright \alpha^3(t) - \alpha^3(x) \blacktriangleright (\alpha^3(y) \cdot (\alpha^3(z) \cdot \alpha^3(t)))$$

$$+\alpha^3(z) \blacktriangleleft (\alpha^3(x) \blacktriangleright (\alpha^3(y) \cdot \alpha^3(t)))$$
$$+\alpha^3([y,z]) \blacktriangleleft (\alpha^3(x) \blacktriangleright \alpha^3(t)) - \alpha^3(y) \blacktriangleleft ((\alpha^3(z) \diamond \alpha^3(x)) \blacktriangleright \alpha^3(t)) = 0.$$

Thus,

$$\begin{aligned}(\alpha(z) \diamond_\alpha (y \diamond_\alpha x)) \blacktriangleright_\alpha \alpha^2(t) - \alpha^2(x) \blacktriangleright_\alpha (\alpha(y) \cdot_\alpha (z \cdot_\alpha t)) + \alpha^2(z) \blacktriangleleft_\alpha (\alpha(x) \blacktriangleright_\alpha (y \cdot_\alpha t))\\ + \alpha([y,z]_\alpha) \blacktriangleleft_\alpha \alpha(x \blacktriangleright_\alpha t) - \alpha^2(y) \blacktriangleleft_\alpha ((z \diamond_\alpha x) \blacktriangleright_\alpha \alpha(t)) = 0.\end{aligned}$$

Hence, $A_\alpha$ is a Hom-M-dendriform algebra.

Moreover, using the multiplicativity of $\alpha$, we obtain

$$\begin{aligned}&(\alpha(z) \diamond_\alpha (y \diamond_\alpha x)) \blacktriangleright_\alpha \alpha^2(t) - \alpha^2(x) \blacktriangleright_\alpha (\alpha(y) \cdot_\alpha (z \cdot_\alpha t))\\ &\quad + \alpha^2(z) \blacktriangleleft_\alpha (\alpha(x) \blacktriangleright_\alpha (y \cdot_\alpha t))\\ &\quad + \alpha([y,z]_\alpha) \blacktriangleleft_\alpha \alpha(x \blacktriangleright_\alpha t) - \alpha^2(y) \blacktriangleleft_\alpha ((z \diamond_\alpha x) \blacktriangleright_\alpha \alpha(t))\\ &= \alpha^3\Big((z \diamond (y \diamond x)) \blacktriangleright t - x \blacktriangleright (y \cdot (z \cdot t)) + z \blacktriangleleft (x \blacktriangleright (y \cdot t))\\ &\quad + [y,z] \blacktriangleleft (x \blacktriangleright t) \ - y \blacktriangleleft ((z \diamond x) \blacktriangleright t)\Big) = 0.\end{aligned}$$

This implies (4.4). Similarly, we obtain identities (4.5)–(4.7).

□

**Remark 4.1.** More generally, let $(A, \blacktriangleright, \blacktriangleleft, \alpha)$ be a Hom-M-dendriform algebra and $\gamma : A \to A$ a Hom-M-dendriform algebra morphism. Define new multiplications on $A$, for all $x, y \in A$, by

$$x \blacktriangleright_\gamma y = \gamma(x) \blacktriangleright \gamma(y) \quad \text{and} \quad x \blacktriangleleft_\gamma y = \gamma(x) \blacktriangleleft \gamma(y).$$

Then $(A, \blacktriangleright_\gamma, \blacktriangleleft_\gamma, \alpha\gamma)$ is a multiplicative Hom-M-dendriform algebra.

**Theorem 4.1.** *Let $(A, \blacktriangleright, \blacktriangleleft, \alpha)$ be a Hom-M-dendriform algebra. Then, the product given by (4.1) (resp. (4.2)) defines a Hom-pre-Malcev algebra $(A, \cdot, \alpha)$ (resp. $(A, \diamond, \alpha)$ ), called the associated horizontal (resp. vertical) Hom-pre-Malcev algebras.*

*Proof.* The identity (2.5) holds since, for any $x, y, z, t \in A$,

$$\begin{aligned}&\alpha([y,z]) \cdot \alpha(x \cdot t) + [[x,y], \alpha(z)] \cdot \alpha^2(t) + \alpha^2(y) \cdot ([x,z] \cdot \alpha(t)) - \alpha^2(x) \cdot (\alpha(y) \cdot (z \cdot t))\\ &\quad + \alpha^2(z) \cdot (\alpha(x) \cdot (y \cdot t))\\ &= \alpha([y,z]) \blacktriangleleft \alpha(x \blacktriangleleft t) + \alpha([y,z]) \blacktriangleleft \alpha(x \blacktriangleright t) + \alpha([y,z]) \blacktriangleright \alpha(x \blacktriangleleft t)\\ &\quad + \alpha([y,z]) \blacktriangleright \alpha(x \blacktriangleright t)\\ &\quad + [[x,y], \alpha(z)] \blacktriangleleft \alpha^2(t) + [[x,y], \alpha(z)] \blacktriangleright \alpha^2(t)\\ &\quad + \alpha^2(y) \blacktriangleleft ([x,z] \blacktriangleleft \alpha(t)) + \alpha^2(y) \blacktriangleleft ([x,z] \blacktriangleright \alpha(t))\\ &\quad + \alpha^2(y) \blacktriangleright ([x,z] \blacktriangleleft \alpha(t)) + \alpha^2(y) \blacktriangleright ([x,z] \blacktriangleright \alpha(t))\\ &\quad + \alpha^2(z) \blacktriangleleft (\alpha(x) \blacktriangleleft (y \blacktriangleleft t))\\ &\quad + \alpha^2(z) \blacktriangleleft (\alpha(x) \blacktriangleleft (y \blacktriangleright t))\\ &\quad + \alpha^2(z) \blacktriangleleft (\alpha(x) \blacktriangleright (y \blacktriangleleft t)) + \alpha^2(z) \blacktriangleleft (\alpha(x) \blacktriangleright (y \blacktriangleright t))\\ &\quad + \alpha^2(z) \blacktriangleright (\alpha(x) \blacktriangleleft (y \blacktriangleleft t))\\ &\quad + \alpha^2(z) \blacktriangleright (\alpha(x) \blacktriangleleft (y \blacktriangleright t))\\ &\quad + \alpha^2(z) \blacktriangleright (\alpha(x) \blacktriangleright (y \blacktriangleleft t)) + \alpha^2(z) \blacktriangleright (\alpha(x) \blacktriangleright (y \blacktriangleright t))\end{aligned}$$

$$
\begin{aligned}
&- \alpha^2(x) \blacktriangleleft (\alpha(y) \blacktriangleleft (z \blacktriangleleft t)) \\
&- \alpha^2(x) \blacktriangleleft (\alpha(y) \blacktriangleleft (z \blacktriangleright t)) \\
&- \alpha^2(x) \blacktriangleleft (\alpha(y) \blacktriangleright (z \blacktriangleleft t)) - \alpha^2(x) \blacktriangleleft (\alpha(y) \blacktriangleright (z \blacktriangleright t)) \\
&- \alpha^2(x) \blacktriangleright (\alpha(y) \blacktriangleleft (z \blacktriangleleft t)) \\
&- \alpha^2(x) \blacktriangleright (\alpha(y) \blacktriangleleft (z \blacktriangleright t)) \\
&- \alpha^2(x) \blacktriangleright (\alpha(y) \blacktriangleright (z \blacktriangleleft t)) \\
&- \alpha^2(x) \blacktriangleright (\alpha(y) \blacktriangleright (z \blacktriangleright t)) \\
= &\alpha([y,z]) \blacktriangleleft \alpha(x \blacktriangleright t) + (\alpha(z) \diamond (y \diamond x)) \blacktriangleright \alpha^2(t) \\
&- \alpha^2(y) \blacktriangleleft ((z \diamond x) \blacktriangleright \alpha(t)) + \alpha^2(z) \blacktriangleleft (\alpha(x) \blacktriangleright (y \cdot t)) \\
&- \alpha^2(x) \blacktriangleright (\alpha(y) \cdot (z \cdot t)) - \alpha(z \diamond y) \blacktriangleright \alpha(x \cdot t) - (\alpha(z) \diamond (x \diamond y)) \blacktriangleright \alpha^2(t) \\
&+ \alpha^2(y) \blacktriangleright ([x,z] \cdot \alpha(t)) \\
&+ \alpha^2(z) \blacktriangleleft (\alpha(x) \blacktriangleleft (y \blacktriangleright t)) - \alpha^2(x) \blacktriangleleft (\alpha(y) \blacktriangleright (z \cdot t)) + \alpha(y \diamond z) \blacktriangleright \alpha(x \cdot t) \\
&+ ([x,y] \diamond \alpha(z)) \blacktriangleright \alpha^2(t) \\
&+ \alpha^2(y) \blacktriangleleft ((x \diamond z) \blacktriangleright \alpha(t)) + \alpha^2(z) \blacktriangleright (\alpha(x) \cdot (y \cdot t)) \\
&- \alpha^2(x) \blacktriangleleft (\alpha(y) \blacktriangleleft (z \blacktriangleright t)) + [[x,y], \alpha(z)] \blacktriangleleft \alpha^2(t) \\
&+ \alpha([y,z]) \blacktriangleleft \alpha(x \blacktriangleleft t) + \alpha^2(y) \blacktriangleleft ([x,z] \blacktriangleleft \alpha(t)) \\
&+ \alpha^2(z) \blacktriangleleft (\alpha(x) \blacktriangleleft (y \blacktriangleleft t)) \\
&- \alpha^2(x) \blacktriangleleft (\alpha(y) \blacktriangleleft (z \blacktriangleleft t)) = 0.
\end{aligned}
$$

□

**Remark 4.2.** The associated horizontal Hom-pre-Malcev algebra and vertical Hom-pre-Malcev algebra, $(A, \cdot, \alpha)$ and $(A, \diamond, \alpha)$, of a Hom-M-dendriform algebra $(A, \blacktriangleright, \blacktriangleleft, \alpha)$ have the same associated Hom-Malcev algebras $(A, [-,-], \alpha)$ defined using (4.3), called the associated Hom-Malcev algebra of the Hom-M-dendriform algebra $(A, \blacktriangleright, \blacktriangleleft, \alpha)$.

Hom-M-dendriform algebras are closely related to the representation of Hom-pre-Malcev algebras.

**Proposition 4.2.** *Let $(A, \blacktriangleright, \blacktriangleleft, \alpha)$ be a Hom-M-dendriform algebra. Let $L_{\blacktriangleleft}$ and $R_{\blacktriangleright}$ be the left and right multiplication operators corresponding respectively to the two operations $\blacktriangleleft$ and $\blacktriangleright$ . Then $(A, L_{\blacktriangleleft}, R_{\blacktriangleright}, \alpha)$ is a representation of its associated horizontal Hom-pre-Malcev algebra $(A, \cdot, \alpha)$.*

*Proof.* If $(A, \blacktriangleright, \blacktriangleleft, \alpha)$ is a Hom-M-dendriform algebra, then for all $x, y \in A$, we have

$$L_{\blacktriangleleft}(\alpha(x))\alpha(y) = \alpha(x) \blacktriangleleft \alpha(y) = \alpha(x \blacktriangleleft y) = \alpha(L_{\blacktriangleleft}(x)y).$$

This implies that $L_{\blacktriangleleft}(\alpha(x))\alpha = \alpha L_{\blacktriangleleft}(x)$. Similarly, we have $R_{\blacktriangleright}(\alpha(x))\alpha = \alpha R_{\blacktriangleright}(x)$.

For any $x, y, z, t \in A$, by (4.7), we have

$$
\begin{aligned}
&L_{\blacktriangleleft}[[x,y], \alpha(z)]\alpha^2(t) - L_{\blacktriangleleft}(\alpha^2(x))L_{\blacktriangleleft}(\alpha(y))L_{\blacktriangleleft}(z)(t) + L_{\blacktriangleleft}(\alpha^2(z))L_{\blacktriangleleft}(\alpha(x))L_{\blacktriangleleft}(y)(t) \\
&\quad + L_{\blacktriangleleft}(\alpha([y,z]))L_{\blacktriangleleft}(\alpha(x))\alpha(t) + L_{\blacktriangleleft}(\alpha^2(y))L_{\blacktriangleleft}([x,z])\alpha(t) \\
&= [[x,y], \alpha(z)] \blacktriangleleft \alpha^2(t) - \alpha^2(x) \blacktriangleleft (\alpha(y) \blacktriangleleft (z \blacktriangleleft t)) + \alpha^2(z) \blacktriangleleft (\alpha(x) \blacktriangleleft (y \blacktriangleleft t))
\end{aligned}
$$

$$+\alpha([y,z])\blacktriangleleft\alpha(x\blacktriangleleft t)+\alpha^2(y)\blacktriangleleft([x,z]\blacktriangleleft\alpha(t))=0.$$

Moreover, by (4.4),

$$\begin{aligned}&R_{\blacktriangleright}(\alpha^2(x))L_\diamond(\alpha(y))L_\diamond(z)(t)-R_{\blacktriangleright}(\alpha(z)\cdot(y\cdot x))\alpha^2(t)+L_{\blacktriangleleft}(\alpha^2(y))R_{\blacktriangleright}(z\cdot x)\alpha(t)\\&\quad+L_{\blacktriangleleft}(\alpha([y,z]))R_{\blacktriangleright}(\alpha(x))\alpha(t)-L_{\blacktriangleleft}(\alpha^2(z))R_{\blacktriangleright}(\alpha(x))L_\diamond(y)(t)\\&=(\alpha(y)\diamond(z\diamond t))\blacktriangleright\alpha^2(x)-\alpha^2(t)\blacktriangleright(\alpha(z)\cdot(y\cdot x))+\alpha^2(y)\blacktriangleleft(\alpha(t)\blacktriangleright(z\cdot x))\\&\quad+\alpha([y,z])\blacktriangleleft\alpha(x\blacktriangleright t)-\alpha^2(z)\blacktriangleleft((y\diamond t)\blacktriangleright\alpha(x))=0.\end{aligned}$$

Similarly, we have

$$\begin{aligned}&R_{\blacktriangleright}(\alpha^2(x))L_\diamond(\alpha(y))L_\diamond(z)(t)-R_{\blacktriangleright}(\alpha(z)\cdot(y\cdot x))\alpha^2(t)\\&\quad+L_{\blacktriangleleft}(\alpha^2(y))R_{\blacktriangleright}(z\cdot x)\alpha(t)+L_{\blacktriangleleft}(\alpha([y,z]))R_{\blacktriangleright}(\alpha(x))\alpha(t)\\&\quad-L_{\blacktriangleleft}(\alpha^2(z))R_{\blacktriangleright}(\alpha(x))L_\diamond(y)(t)=0,\\&L_{\blacktriangleleft}(\alpha^2(y))L_{\blacktriangleleft}(\alpha(z))R_{\blacktriangleright}(x)(t)-R_{\blacktriangleright}(\alpha^2(x))L_\diamond(\alpha(y))L_\diamond(z)(t)\\&\quad-L_{\blacktriangleleft}(\alpha^2(z))R_{\blacktriangleright}(y\cdot x)\alpha(t)-R_{\blacktriangleright}(\alpha(z\cdot x))L_\diamond(\alpha(y))\alpha(t)\\&\quad+R_{\blacktriangleright}([z,y]\cdot\alpha(x))\alpha^2(t)=0.\end{aligned}$$

Thus, $(A,L_{\blacktriangleleft},R_{\blacktriangleright},\alpha)$ is a representation of the Hom-pre-Malcev algebra $(A,\cdot,\alpha)$.

□

**Corollary 4.1.** *Let $(A,\blacktriangleright,\blacktriangleleft,\alpha)$ be a regular Hom-M-dendriform algebra and $A^*$ the dual of $A$. Then $(A^*,L^\star_{\blacktriangleleft}-R^\star_{\blacktriangleright},-R^\star_{\blacktriangleright},(\alpha^{-1})^*)$ is a representation of the associated horizontal Hom-pre-Malcev algebra $(A,\cdot,\alpha)$ and $(A^*,L^\star_{\blacktriangleright}+L^\star_{\blacktriangleleft},L^\star_{\blacktriangleleft},(\alpha^{-1})^*)$ is a representation of the associated vertical Hom-pre-Malcev algebra $(A,\diamond,\alpha)$.*

*Proof.* This follows from Theorem 3.1 and Proposition 4.2.

□

**Proposition 4.3.** *Let $(A,\blacktriangleright,\blacktriangleleft,\alpha)$ be a Hom-M-dendriform algebra. With two binary operations $\blacktriangleright^t,\blacktriangleleft^t\colon A\otimes A\to A$ defined for all $x,y\in A$ by*

$$x\blacktriangleright^t y=-y\blacktriangleright x,\quad x\blacktriangleleft^t y=x\blacktriangleleft y,\tag{4.8}$$

*$(A,\blacktriangleright^t,\blacktriangleleft^t,\alpha)$ is a Hom-M-dendriform algebra.*

*Its associated horizontal Hom-pre-Malcev algebra is the associated vertical Hom-pre-Malcev algebra $(A,\diamond,\alpha)$ of $(A,\blacktriangleright,\blacktriangleleft,\alpha)$, and its associated vertical Hom-pre-Malcev algebra is the associated horizontal Hom-pre-Malcev algebra $(A,\cdot,\alpha)$ of $(A,\blacktriangleright,\blacktriangleleft,\alpha)$, that is,*

$$x\cdot^t y=x\blacktriangleleft^t y+x\blacktriangleright^t y=x\blacktriangleleft y-y\blacktriangleright x=x\diamond y,\tag{4.9}$$

$$x\diamond^t y=x\blacktriangleleft^t y-y\blacktriangleright^t x=x\blacktriangleleft y+x\blacktriangleright y=x\cdot y,\tag{4.10}$$

$$\begin{aligned}[x,y]^t&=x\blacktriangleleft^t y+x\blacktriangleright^t y-y\blacktriangleleft^t x-y\blacktriangleright^t x\\&=x\blacktriangleleft y-y\blacktriangleright x-y\blacktriangleleft x+x\blacktriangleright y=[x,y].\end{aligned}\tag{4.11}$$

*Proof.* By (4.9)–(4.11), for all $x,y,z,t\in A$,

$$\begin{aligned}&[[x,y]^t,\alpha(z)]^t\blacktriangleleft^t\alpha^2(t)-\alpha^2(x)\blacktriangleleft^t(\alpha(y)\blacktriangleleft^t(z\blacktriangleleft^t t))+\alpha^2(z)\blacktriangleleft^t(\alpha(x)\blacktriangleleft^t(y\blacktriangleleft^t t))\\&\quad+\alpha([y,z]^t)\blacktriangleleft^t\alpha(x\blacktriangleleft^t t)+\alpha^2(y)\blacktriangleleft^t([x,z]^t\blacktriangleleft^t\alpha(t))\end{aligned}$$

$$= [[x,y],\alpha(z)] \blacktriangleleft \alpha^2(t) - \alpha^2(x) \blacktriangleleft (\alpha(y) \blacktriangleleft (z \blacktriangleleft t)) + \alpha^2(z) \blacktriangleleft (\alpha(x) \blacktriangleleft (y \blacktriangleleft t))$$
$$+ \alpha([y,z]) \blacktriangleleft \alpha(x \blacktriangleleft t) + \alpha^2(y) \blacktriangleleft ([x,z] \blacktriangleleft \alpha(t)).$$

Similarly, (4.4)$^t$=(4.4), (4.5)$^t$=(4.5), and (4.6)$^t$=(4.6). Thus, $(A, \blacktriangleright^t, \blacktriangleleft^t, \alpha)$ is a Hom-M-dendriform algebra.

□

**Definition 4.2.** Let $(A, \blacktriangleright, \blacktriangleleft, \alpha)$ be a Hom-M-dendriform algebra. The Hom-M-dendriform algebra $(A, \blacktriangleright^t, \blacktriangleleft^t, \alpha)$ given by (4.8) is called the transpose of $(A, \blacktriangleright, \blacktriangleleft, \alpha)$.

### 4.2. Hom-M-Dendriform Algebras and $\mathcal{O}$-Operators of Hom-Pre-Malcev Algebras

Examples of Hom-M-dendriform algebras can be constructed from Hom-pre-Malcev algebras with $\mathcal{O}$-operators. Briefly, we only give the study involving the associated horizontal Hom-pre-Malcev algebras.

**Proposition 4.4.** *Let $(A, \cdot, \alpha)$ be a Hom-pre-Malcev algebra and $(V, \ell, r, \beta)$ be a representation of $A$. Let $T$ be an $\mathcal{O}$-operator of $(A, \cdot, \alpha)$ associated with $(V, \ell, r, \beta)$. Then $(V, \blacktriangleright, \blacktriangleleft, \beta)$ is a Hom-M-dendriform algebra, where for all $a, b \in V$,*

$$a \blacktriangleright b = r(T(b))a, \quad a \blacktriangleleft b = \ell(T(a))b. \tag{4.12}$$

*Therefore, there is a Hom-pre-Malcev algebra on $V$ given in Theorem 4.1 as the associated horizontal Hom-pre-Malcev algebra of $(V, \blacktriangleright, \blacktriangleleft, \beta)$, and $T$ is a morphism of Hom-pre-Malcev algebras. Moreover, $T(V) = \{T(v) \mid v \in V\} \subseteq A$ is a Hom-pre-Malcev subalgebra of $(A, \cdot, \alpha)$, and there is an induced Hom-M-dendriform algebraic structure on $(T(V), \triangleright, \triangleleft, \alpha)$ given for all $a, b \in V$ by*

$$T(a) \triangleright T(b) = T(a \blacktriangleright b), \quad T(a) \triangleleft T(b) = T(a \blacktriangleleft b).$$

*Its corresponding associated horizontal Hom-pre-Malcev algebraic structure on $T(V)$ is just the subalgebra of the Hom-pre-Malcev $(A, \cdot, \alpha)$, and $T$ is a homomorphism of Hom-M-dendriform algebras.*

*Proof.* For any $a, b \in V$, define $\cdot_V, \diamond_V, [-,-]_V : \otimes^3 V \to V$ by

$$a \cdot_V b = a \blacktriangleleft b + a \blacktriangleright b,$$
$$a \diamond_V b = a \blacktriangleleft b - b \blacktriangleright a,$$
$$[a,b]_V = a \cdot_V b - b \cdot_V a = a \diamond_V b - b \diamond_V a.$$

Using identity (3.2), we have

$$T(a \cdot_V b) = T(a \blacktriangleleft b + a \blacktriangleright b) = T(\ell(T(a))b + r(T(b))a) = T(a) \cdot T(b),$$
$$T([a,b]_V) = T(a \cdot_V b - b \cdot_V a) = T(a) \cdot T(b) - T(b) \cdot T(a) = [T(a), T(b)].$$

Hence, for any $a, b, c, d \in V$, we have

$$(\beta(c) \diamond_V (b \diamond_V a)) \blacktriangleright \beta^2(d) - \beta^2(a) \blacktriangleright (\beta(b) \cdot_V (c \cdot_V d))$$
$$+ \beta^2(c) \blacktriangleleft (\beta(a) \blacktriangleright (b \cdot_V d))$$
$$+ \beta([b,c]_V) \blacktriangleleft \beta(a \blacktriangleright d) - \beta^2(b) \blacktriangleleft ((c \diamond_V a) \blacktriangleright \beta(d))$$
$$= r(T(\beta^2(d)))(\beta(c) \diamond_V (b \diamond_V a)) - r(T(\beta(b) \cdot_V (c \cdot_V d)))\beta^2(a)$$

$$
\begin{aligned}
&+ \ell(T(\beta^2(c)))r(T(b \cdot_V d))\beta(a)\\
&+ \ell(T(\beta([b,c]_V)))\beta(r(T(d))a) - \ell(T(\beta^2(b)))r(T(\beta(d)))(c \diamond_V a)\\
&= r(\alpha^2(T(d)))\rho(\alpha(T(c)))\rho(T(b))a - r(\alpha(T(b)) \cdot (T(c) \cdot T(d)))\beta^2(a)\\
&+ \ell(\alpha^2(T(c)))r(T(b) \cdot T(d))\beta(a)\\
&+ \ell(\alpha([T(b), T(c)]))r(\alpha(T(d)))\beta(a) - \ell(\alpha^2(T(b)))r(\alpha(T(d)))\rho(T(c))a = 0.
\end{aligned}
$$

This implies that (4.4) holds. Moreover, (4.5) holds. Indeed,

$$
\begin{aligned}
&\beta^2(c) \blacktriangleleft (\beta(a) \blacktriangleleft (b \blacktriangleright d)) - (\beta(c) \diamond_V (a \diamond_V b)) \blacktriangleright \beta^2(d)\\
&\quad - \beta^2(a) \blacktriangleleft (\beta(b) \blacktriangleright (c \cdot_V d))\\
&\quad - \beta(c \diamond_V b) \blacktriangleright \beta(a \cdot_V d) + \beta^2(b) \blacktriangleright ([a,c]_V \cdot_V \beta(d))\\
&= \ell(T(\beta^2(c)))\ell(T(\beta(a))r(T(d))b - r(T(\beta^2(d))(\beta(c) \diamond_V (a \diamond_V b))\\
&\quad - \ell(T(\beta^2(a)))r(T(c \cdot_V d))\beta(b)\\
&\quad - r(T(\beta(a \cdot_V b)))\beta(c \diamond_V b) + r(T([a,c]_V \cdot_V \beta(x)))\beta^2(b)\\
&= \ell(\alpha^2(T(c)))\ell(\alpha(T(a)))r(T(d))b - r(\alpha^2(T(d)))\rho(\alpha(T(c)))\rho(T(a))b\\
&\quad - \ell(\alpha^2(T(a)))r(T(c) \cdot T(d))\beta(b)\\
&\quad - r(\alpha(T(a) \cdot T(b)))\rho(\alpha(T(c))\beta(b) + r([T(a), T(c)] \cdot \alpha(T(d)))\beta^2(b) = 0.
\end{aligned}
$$

To prove identity (4.6), we compute as follows

$$
\begin{aligned}
&\beta^2(c) \blacktriangleright (\beta(a) \cdot_V (b \cdot_V d)) + ([a,b]_V \diamond_V \beta(c)) \blacktriangleright \beta^2(d)\\
&\quad - \beta^2(a) \blacktriangleleft (\beta(b) \blacktriangleleft (c \blacktriangleright d))\\
&\quad + \beta(b \diamond_V c) \blacktriangleright \beta(a \cdot_V d) + \beta^2(b) \blacktriangleleft ((a \diamond_V c) \blacktriangleright \beta(d))\\
&= r(\alpha(T(a)) \cdot (T(b) \cdot T(d)))\beta^2(c) + r(\alpha^2(T(d)))\rho([T(a), T(b)])\beta(c)\\
&\quad - \ell(\alpha^2(T(a)))\ell(\alpha(T(b)))r(T(d))c\\
&\quad + r(\alpha(T(a) \cdot T(d)))\rho(\alpha(T(b)))\beta(c) + \ell(\alpha^2(T(b)))r(\alpha(T(d)))\rho(T(a))c = 0.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&[[a,b]_V, \beta(c)]_V \blacktriangleleft \beta^2(d) - \beta^2(a) \blacktriangleleft (\beta(b) \blacktriangleleft (c \blacktriangleleft d))\\
&\quad + \beta^2(c) \blacktriangleleft (\beta(a) \blacktriangleleft (b \blacktriangleleft d))\\
&\quad + \beta([b,c]_V) \blacktriangleleft \beta(a \blacktriangleleft d) + \beta^2(b) \blacktriangleleft ([a,c]_V \blacktriangleleft \beta(d))\\
&= \ell([[T(a), T(b)], \alpha(T(c))])\beta^2(d) - \ell(\alpha^2(T(a)))\ell(\alpha(T(b)))\ell(T(c))\\
&\quad + \ell(\alpha^2(T(c)))\ell(\alpha(T(a)))\ell(T(b))d\\
&\quad + \ell(\alpha([T(b), T(c)])\ell(\alpha(T(a)))\beta(d) + \ell(\alpha^2(T(b)))\ell([T(a), T(c)])\beta(d) = 0.
\end{aligned}
$$

□

Now, we introduce the following concept of the Rota–Baxter operator on a Hom-pre-Malcev algebra which is a particular case of $\mathcal{O}$-operator associated with the regular representation.

**Definition 4.3.** Let $(A, \cdot, \alpha)$ be a Hom-pre-Malcev algebra. A linear map $\mathcal{R} : A \longrightarrow A$ is called a Rota–Baxter operator of weight zero on $A$ if for all $x, y \in A$,

$$\mathcal{R} \circ \alpha = \alpha \circ \mathcal{R}, \qquad \mathcal{R}(x) \cdot \mathcal{R}(y) = \mathcal{R}\big(\mathcal{R}(x) \cdot y + x \cdot \mathcal{R}(y)\big).$$

**Corollary 4.2.** *Let $(A, \cdot, \alpha)$ be a Hom-pre-Malcev algebra and $\mathcal{R} : A \to A$ a Rota–Baxter operator of weight 0 for $A$. Define new operations on $A$ by*

$$x \blacktriangleright y = x \cdot \mathcal{R}(y), \qquad x \blacktriangleleft y = \mathcal{R}(x) \cdot y.$$

*Then, $(A, \blacktriangleright, \blacktriangleleft, \alpha)$ is a Hom-M-dendriform algebra.*

**Theorem 4.2.** *Let $(A, \cdot, \alpha)$ be a Hom-pre-Malcev algebra. Then, there exists a compatible Hom-M-dendriform algebra if and only if there exists an invertible $\mathcal{O}$-operator on $A$ associated with a representation $(V, \ell, r, \beta)$.*

*Proof.* Let $T$ be an invertible $\mathcal{O}$-operator of $A$ associated with $(V, \ell, r, \beta)$. Then, using (4.12), there is a Hom-M-dendriform algebra structure on $T(V) = A$ given for all $a, b \in V$ by

$$T(a) \blacktriangleright T(b) = T(r(T(b))a), \quad T(a) \blacktriangleleft T(b) = T(\ell(T(a))b).$$

If we put $x = T(a)$ and $y = T(b)$, then we get

$$x \blacktriangleright y = T(r(y)T^{-1}(x)), \quad x \blacktriangleleft y = T(\ell(x)T^{-1}(y)).$$

This is a compatible Hom-M-dendriform algebra structure on $A$. Indeed,

$$x \blacktriangleleft y + x \blacktriangleright y = T(\ell(x)T^{-1}(y)) + T(r(y)T^{-1}(x)),$$
$$TT^{-1}(x) \cdot TT^{-1}(y) = x \cdot y.$$

Conversely, let $(A, \blacktriangleright, \blacktriangleleft, \alpha)$ be a Hom-M-dendriform algebra and $(A, \cdot, \alpha)$ be the associated horizontal Hom-pre-Malcev algebra. Then $(A, L_{\blacktriangleleft}, R_{\blacktriangleright}, \alpha)$ is a representation of $(A, \cdot, \alpha)$ and the identity map $id : A \to A$ is an invertible $\mathcal{O}$-operator of $(A, \cdot, \alpha)$ associated to $(A, L_{\blacktriangleleft}, R_{\blacktriangleright}, \alpha)$.

□

**Lemma 4.1.** *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two commuting Rota–Baxter operators (of weight zero) on a Hom-Malcev algebra $(A, [-,-], \alpha)$. Then $\mathcal{R}_2$ is a Rota–Baxter operator (of weight zero) on the Hom-pre-Malcev algebra $(A, \cdot, \alpha)$, where for all $x, y \in A$,*

$$x \cdot y = [\mathcal{R}_1(x), y].$$

*Proof.* For any $x, y \in A$,

$$\begin{aligned}\mathcal{R}_2(x) \cdot \mathcal{R}_2(y) = [\mathcal{R}_1(\mathcal{R}_2(x)), \mathcal{R}_2(y)] &= \mathcal{R}_2([\mathcal{R}_1(\mathcal{R}_2(x)), y] + [\mathcal{R}_1(x), \mathcal{R}_2(y)]) \\ &= \mathcal{R}_2(\mathcal{R}_2(x) \cdot y + x \cdot \mathcal{R}_2(y)).\end{aligned}$$

□

**Corollary 4.3.** *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two commuting Rota–Baxter operators (of weight zero) on a Hom-Malcev algebra $(A, [-,-], \alpha)$. Then there exists a Hom-M-dendriform algebra structure on $A$ given for all $x, y \in A$ by*

$$x \blacktriangleright y = [\mathcal{R}_1(x), \mathcal{R}_2(y)], \quad x \blacktriangleleft y = [\mathcal{R}_1(\mathcal{R}_2(x)), y]. \tag{4.13}$$

*Proof.* By Lemma 4.1, $\mathcal{R}_2$ is a Rota–Baxter operator of weight zero on $(A, \cdot, \alpha)$, where

$$x \cdot y = [\mathcal{R}_1(x), y].$$

Then, applying Corollary 4.2, there exists a Hom-M-dendriform algebraic structure on $A$ given for all $x, y \in A$ by $x \blacktriangleright y = x \cdot \mathcal{R}_2(y) = [\mathcal{R}_1(x), \mathcal{R}_2(y)]$ and $x \blacktriangleleft y = \mathcal{R}_2(x) \cdot y = [\mathcal{R}_1(\mathcal{R}_2(x)), y]$.

□

**Example 4.1.** Consider the 4-dimensional Hom-Malcev algebra and the Rota–Baxter operators $\mathcal{R}$ given in [23, Example 3.2]. Then, there is a Hom-M-dendriform algebraic structure on $A$ given for all $x, y \in A$ by $x \blacktriangleright_\alpha y = \alpha([\mathcal{R}(x), \mathcal{R}(y)])$ and $x \blacktriangleleft_\alpha y = \alpha([\mathcal{R}^2(x), y])$, that is

| $\blacktriangleright_\alpha$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $e_1$ | $0$ | $\lambda_1 e_3$ | $0$ | $0$ |
| $e_2$ | $-\lambda_1 e_3$ | $0$ | $0$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $0$ |
| $e_4$ | $0$ | $0$ | $0$ | $0$ |

| $\blacktriangleleft_\alpha$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ |
|---|---|---|---|---|
| $e_1$ | $\frac{a_4}{2} e_4$ | $-\alpha(e_2)$ | $e_3$ | $-e_4$ |
| $e_2$ | $0$ | $0$ | $0$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $0$ |
| $e_4$ | $0$ | $0$ | $0$ | $0$ |

.

**Example 4.2.** Consider the 5-dimensional Hom-Malcev algebra and Rota–Baxter operators $\mathcal{R}$ given in [23, Example 3.3]. Then, there is a Hom-M-dendriform algebraic structure on $A$ given for all $x, y \in A$ by $x \blacktriangleright_\alpha y = \alpha([\mathcal{R}(x), \mathcal{R}(y)])$, $x \blacktriangleleft_\alpha y = \alpha([\mathcal{R}^2(x), y])$, that is

| $\blacktriangleright_\alpha$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|---|---|---|---|---|---|
| $e_1$ | $0$ | $0$ | $0$ | $be_3$ | $0$ |
| $e_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_4$ | $-be_3$ | $0$ | $0$ | $0$ | $0$ |
| $e_5$ | $0$ | $0$ | $0$ | $0$ | $0$ |

| $\blacktriangleleft_\alpha$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ |
|---|---|---|---|---|---|
| $e_1$ | $-a_4 e_2$ | $-a_5 e_3$ | $0$ | $e_2$ | $\frac{-ba_4}{a_5} e_3$ |
| $e_2$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_3$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_4$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $e_5$ | $0$ | $0$ | $0$ | $0$ | $0$ |

.

In the sequel, we provide the relation between Hessian structures and Hom-M-dendriform algebras.

**Definition 4.4.** A Hessian structure on a regular Hom-pre-Malcev algebra is a symmetric nondegenerate bilinear form satisfying, for all $x, y, z \in A$,

$$\mathcal{B}(\alpha(x), \alpha(y)) = \mathcal{B}(x, y), \tag{4.14}$$

$$\mathcal{B}(x \cdot y, \alpha(z)) - \mathcal{B}(\alpha(x), y \cdot z) = \mathcal{B}(y \cdot x, \alpha(z)) - \mathcal{B}(\alpha(y), x \cdot z). \tag{4.15}$$

**Proposition 4.5.** *Let $(A, \cdot, \alpha)$ be a regular Hom-pre-Malcev algebra with a Hessian structure $\mathcal{B}$, and $(A^*, ad^\star, -R_\cdot^\star, (\alpha^{-1})^*)$ be a representation of $(A, \cdot, \alpha)$. Then there exists a compatible Hom-M-dendriform algebra structure on $(A, \cdot, \alpha)$ given for all $x, y, z \in A$ by*

$$\mathcal{B}(x \blacktriangleright y, \alpha(z)) = \mathcal{B}(\alpha(x), z \cdot y), \qquad \mathcal{B}(x \blacktriangleleft y, \alpha(z)) = -\mathcal{B}(\alpha(y), [x, z]). \tag{4.16}$$

*Proof.* Define the linear map $T : A^* \to A$ by $\langle T^{-1}(x), y\rangle = \mathcal{B}(x, y)$. Since $\mathcal{B}$ is $\alpha$-symmetric and using (4.15), we obtain that $T$ is an invertible $\mathcal{O}$-operator on $A$ associated to the representation $(A^*, ad^\star, -R_\cdot^\star, (\alpha^{-1})^*)$. By Corollary 4.2, there exists a compatible Hom-M-dendriform algebra structure given by $x \blacktriangleright y = -T(R_\cdot^\star(y)T^{-1}(x)), \quad x \blacktriangleleft y = T(ad^\star(x)T^{-1}(y))$. Hence,

$$\begin{aligned}
\mathcal{B}(x \blacktriangleright y, \alpha(z)) &= -\big\langle T^{-1}(x \blacktriangleright y), \alpha(z)\big\rangle \\
&= -\big\langle R_\cdot^\star(y)T^{-1}(x), \alpha(z)\big\rangle = \big\langle T^{-1}(x), \alpha(z)\cdot\alpha(y)\big\rangle \\
&= \mathcal{B}(x, \alpha(z)\cdot\alpha(y)) = \mathcal{B}(x, \alpha(z\cdot y)) = \mathcal{B}(\alpha(x), \alpha^2(z\cdot y)) = \mathcal{B}(\alpha(x), z\cdot y), \\
\mathcal{B}(x \blacktriangleleft y, \alpha(z)) &= \big\langle T^{-1}(x \blacktriangleleft y), \alpha(z)\big\rangle \\
&= \big\langle ad^\star(x)T^{-1}(y), \alpha(z)\big\rangle = -\big\langle T^{-1}(y), [\alpha(x), \alpha(z)]\big\rangle \\
&= -\mathcal{B}(y, [\alpha(x), \alpha(z)]) = -\mathcal{B}(y, \alpha([x, z])) = -\mathcal{B}(\alpha(y), \alpha^2([x, z])) \\
&= -\mathcal{B}(\alpha(y), [x, z]).
\end{aligned}$$

□

## 5. Hom-M-Dendriform Algebras Starting from Hom-Alternative Quadri-algebras

Hom-M-dendriform algebras are related to Hom-alternative quadri-algebras in the same way as Hom-L-dendriform algebras are related to Hom-quadri-algebras (see [15] for more details).

**Definition 5.1.** Let $(A, *, \alpha)$ be a Hom-alternative algebra and $V$ a vector space. Let $\mathfrak{l}, \mathfrak{r} : A \to End(V)$ and $\beta \in End(V)$ be linear maps. Then $(V, \mathfrak{l}, \mathfrak{r}, \beta)$ is called a bimodule, or a representation, of $(A, *, \alpha)$, if for any $x, y \in A$,

$$\mathfrak{l}(x^2)\beta = \mathfrak{l}(\alpha(x))\mathfrak{l}(x), \quad \mathfrak{r}(x^2)\beta = \mathfrak{r}(\alpha(x))\mathfrak{r}(x), \tag{5.1}$$

$$\mathfrak{r}(\alpha(y))\mathfrak{l}(x) - \mathfrak{l}(\alpha(x))\mathfrak{r}(y) = \mathfrak{r}(x * y)\beta - \mathfrak{r}(\alpha(y))\mathfrak{r}(x), \tag{5.2}$$

$$\mathfrak{l}(y * x)\beta - \mathfrak{l}(\alpha(y))\mathfrak{l}(x) = \mathfrak{l}(\alpha(y))\mathfrak{r}(x) - \mathfrak{r}(\alpha(x))\mathfrak{l}(y). \tag{5.3}$$

In the following, we recall the notion of representation of Hom-pre-alternative algebras introduced in [51].

**Definition 5.2.** *([51])* Let $(A, \prec, \succ, \alpha)$ be a Hom-pre-alternative algebra and $(V, \beta)$ a vector space. Let $L_\succ, R_\succ, L_\prec, , R_\prec : A \to End(V)$ be linear maps. Then, $(V, L_\succ, R_\succ, L_\prec, , R_\prec, \beta)$ ia called a representation of $(A \prec, \succ, \alpha)$ if for any $x, y \in A$, and $x * y = x \prec y + y \succ x$, $L = L_\prec + L_\succ$ and $R = R_\prec + R_\succ$,

$$L_\succ(x * y + y * x)\beta = L_\succ(\alpha(x))L_\succ(y) + L_\succ(\alpha(y))L_\succ(x), \tag{5.4}$$

$$R_\succ(\alpha(y))(L(x) + R(x)) = L_\succ(\alpha(x))R_\succ(y) + R_\succ(x \succ y)\beta, \tag{5.5}$$

$$R_\prec(\alpha(y))L_\succ(x) + R_\prec(\alpha(y))R_\prec(x) = L_\succ(\alpha(x))R_\prec(y) + R_\prec(x * y)\beta, \tag{5.6}$$

$$R_\prec(\alpha(y))R_\succ(x) + R_\succ(\alpha(y))L_\prec(x) = L_\prec(\alpha(x))R(y) + R_\succ(x * y)\beta, \tag{5.7}$$

$$L_\prec(y \prec x)\beta + L_\prec(x \succ y)\beta = L_\prec(\alpha(y))L(x) + L_\succ(\alpha(y))L_\succ(x), \tag{5.8}$$

$$R_\prec(\alpha(x))L_\succ(y) + L_\succ(y \succ x)\beta = L_\succ(y)R_\prec(x) + L_\succ(\alpha(y))L_\succ(x), \tag{5.9}$$

$$R_\prec(\alpha(x))R_\succ(y) + R_\succ(\alpha(y))R(x) = R_\succ(y \prec x)\beta + R_\succ(x \succ y)\beta, \tag{5.10}$$

$$L_\prec(y \succ x)\beta + R_\succ(\alpha(x))L(y) = L_\succ(\alpha(y))L_\prec(x) + L_\succ(\alpha(y))R_\succ(y), \tag{5.11}$$

$$R_{\prec}(\alpha(x))R_{\prec}(y) + R_{\prec}(\alpha(y))R_{\prec}(x) = R_{\prec}(x * y + y * x)\beta, \tag{5.12}$$

$$R_{\prec}(\alpha(y))L_{\prec}(x) + L_{\prec}(x \prec y)\beta = L_{\prec}(\alpha(x))(R(y) + L(y)). \tag{5.13}$$

**Proposition 5.1.** *A tuple $(V, L_{\succ}, R_{\succ}, L_{\prec}, R_{\prec}, \beta)$ is a representation of a Hom-pre-alternative algebra $(A, \prec, \succ, \alpha)$ if and only if the direct sum $(A \oplus V, \ll, \gg, \alpha + \beta)$ is a Hom-pre-alternative algebra, where for any $x, y \in A, a, b \in V$,*

$$\begin{aligned}(x + a) \ll (y + b) &= x \prec y + L_{\prec}(x)b + R_{\prec}(y)a,\\ (x + a) \gg (y + b) &= x \succ y + L_{\succ}(x)b + R_{\succ}(x)a,\\ (\alpha \oplus \beta)(x + a) &= \alpha(x) + \beta(a).\end{aligned}$$

*We denote it by $A \ltimes^{\alpha,\beta}_{L_{\succ},R_{\succ},L_{\prec},R_{\prec}} V$ or simply $A \ltimes V$.*

**Definition 5.3.** Let $(V, \mathfrak{l}, \mathfrak{r}, \beta)$ be a representation of a Hom-alternative algebra $(A, *, \alpha)$. An $\mathcal{O}$-operator of Hom-alternative algebra $(A, *, \alpha)$ with respect to the representation $(V, \mathfrak{l}, \mathfrak{r}, \beta)$ is a linear map $T : V \to A$ such that, for all $a, b \in V$,

$$\alpha \circ T = T \circ \beta \text{ and } T(a) * T(b) = T\big(\mathfrak{l}(T(a))b + \mathfrak{r}(T(b))a\big). \tag{5.14}$$

**Remark 5.1.** Rota–Baxter operator of weight 0 on a Hom-alternative algebra $(A, *, \alpha)$ is an $\mathcal{O}$-operator associated to the representation $(A, L, R, \alpha)$, where $L$ and $R$ are the left and right multiplication operators corresponding to the multiplication $*$.

Recall from [43] that a Hom-algebra $(A, [-,-], \alpha)$ is said to be a Hom-Malcev admissible algebra if, for any elements $x, y \in A$, the bracket $[-,-] : A \times A \to A$ defined by $[x, y] = x * y - y * x$ satisfies the Hom-Malcev identity.

**Proposition 5.2.** [23] *Let $T : V \to A$ be an $\mathcal{O}$-operator of the Hom-alternative algebra $(A, *, \alpha)$ with respect to the representation $(V, \mathfrak{l}, \mathfrak{r}, \beta)$. Then $(V, \prec, \succ, \beta)$ be a Hom-pre-alternative algebra, where for all $a, b \in V$,*

$$a \succ b = \mathfrak{l}(T(a))b \quad \text{and} \quad a \prec b = \mathfrak{r}(T(b))a. \tag{5.15}$$

*Moreover, if $(V, \cdot, \beta)$ is the multiplicative Hom-pre-Malcev algebra associated to the Hom-Malcev admissible algebra $(A, [-,-], \alpha)$ on the representation $(V, \mathfrak{l} - \mathfrak{r}, \beta)$, then $a \cdot b = a \succ b - b \prec a$.*

**Corollary 5.1.** [23] *Let $(A, *, \alpha)$ be a Hom-alternative algebra and $\mathcal{R} : A \to A$ be a Rota–Baxter operator of weight 0 such that $\mathcal{R}\alpha = \alpha\mathcal{R}$. If multiplications $\prec$ and $\succ$ on $A$ are defined for all $x, y \in A$ by $x \prec y = x * \mathcal{R}(y)$ and $x \succ y = \mathcal{R}(x) * y$, then $(A, \prec, \succ, \alpha)$ is a Hom-pre-alternative algebra.*

*Moreover, if $(A, \cdot, \alpha)$ is the multiplicative Hom-pre-Malcev algebra associated with the Hom-Malcev admissible algebra $(A, [-,-], \alpha)$, then $x \cdot y = x \succ y - y \prec x$.*

Now, we introduce the Hom version of the alternative quadri-algebras.

**Definition 5.4.** A **Hom-alternative quadri-algebra** is a 6-tuple $(A, \nwarrow, \swarrow, \nearrow, \searrow, \alpha)$ consisting of a vector space $A$, four bilinear maps $\nwarrow, \swarrow, \nearrow, \searrow : A \times$

$A \to A$ and a linear map $\alpha : A \to A$ which is algebra morphism such that the following axioms are satisfied for all $x, y, z \in A$ :

$$\begin{aligned}
((x,y,z))^r_\alpha + ((y,x,z))^m_\alpha &= 0, & ((x,y,z))^r_\alpha + ((x,z,y))^r_\alpha &= 0,\\
((x,y,z))^n_\alpha + ((y,x,z))^w_\alpha &= 0, & ((x,y,z))^n_\alpha + ((x,z,y))^{ne}_\alpha &= 0,\\
((x,y,z))^{ne}_\alpha + ((y,x,z))^e_\alpha &= 0, & ((x,y,z))^w_\alpha + ((x,z,y))^{sw}_\alpha &= 0,\\
((x,y,z))^{sw}_\alpha + ((y,x,z))^s_\alpha &= 0, & ((x,y,z))^m_\alpha + ((x,z,y))^\ell_\alpha &= 0,\\
((x,y,z))^\ell_\alpha + ((y,x,z))^\ell_\alpha &= 0,
\end{aligned}$$

where

$$\begin{aligned}
((x,y,z))^r_\alpha &= (x \nwarrow y) \nwarrow \alpha(z) - \alpha(x) \nwarrow (y * z) && \text{(right } \alpha\text{-associator)}\\
((x,y,z))^\ell_\alpha &= (x * y) \searrow \alpha(z) - \alpha(x) \searrow (y \searrow z) && \text{(left } \alpha\text{-associator)}\\
((x,y,z))^m_\alpha &= (x \searrow y) \nwarrow \alpha(z) - \alpha(x) \searrow (y \nwarrow z) && \text{(middle } \alpha\text{-associator)}\\
((x,y,z))^n_\alpha &= (x \nearrow y) \nwarrow \alpha(z) - \alpha(x) \nearrow (y \prec z) && \text{(north } \alpha\text{-associator)}\\
((x,y,z))^w_\alpha &= (x \swarrow y) \nwarrow \alpha(z) - \alpha(x) \swarrow (y \wedge z) && \text{(west } \alpha\text{-associator)}\\
((x,y,z))^s_\alpha &= (x \succ y) \swarrow \alpha(z) - \alpha(x) \searrow (y \swarrow z) && \text{(south } \alpha\text{-associator)}\\
((x,y,z))^e_\alpha &= (x \vee y) \nearrow \alpha(z) - \alpha(x) \searrow (y \nearrow z) && \text{(east } \alpha\text{-associator)}\\
((x,y,z))^{ne}_\alpha &= (x \wedge y) \nearrow \alpha(z) - \alpha(x) \nearrow (y \succ z) && \text{(north-east } \alpha\text{-associator)}\\
((x,y,z))^{sw}_\alpha &= (x \prec y) \swarrow \alpha(z) - \alpha(x) \swarrow (y \vee z) && \text{(south-west } \alpha\text{-associator)}
\end{aligned}$$

$$\begin{aligned}
x \succ y &= x \nearrow y + x \searrow y, & x \prec y &= x \nwarrow y + x \swarrow y,\\
x \vee y &= x \searrow y + x \swarrow y, & x \wedge y &= x \nearrow y + x \nwarrow y,\\
x * y &= x \succ y + x \prec y \ = x \searrow y + x \nearrow y + x \nwarrow y + x \swarrow y.
\end{aligned}$$

**Lemma 5.1.** *Let $(A, \nearrow, \searrow, \swarrow, \nwarrow, \alpha)$ be a Hom-alternative quadri-algebra. Then $(A, \prec, \succ, \alpha)$ and $(A, \vee, \wedge, \alpha)$ are Hom-pre-alternative algebras (called respectively horizontal and vertical Hom-pre-alternative structures associated to $A$), and $(A, *, \alpha)$ is a Hom-alternative algebra.*

A morphism $f : (A, \nwarrow, \swarrow, \nearrow, \searrow, \alpha) \to (A', \nwarrow', \swarrow', \nearrow', \searrow', \alpha')$ of Hom-alternative quadri-algebras is a linear map $f : A \to A'$ satisfying $f(x \nearrow y) = f(x) \nearrow' f(y), f(x \searrow y) = f(x) \searrow' f(y), f(x \nwarrow y) = f(x) \nwarrow' f(y)$ and $f(x \swarrow y) = f(x) \swarrow' f(y)$, for all $x, y \in A$, as well as $f\alpha = \alpha' f$.

**Proposition 5.3.** *Let $(A, \nwarrow, \swarrow, \nearrow, \searrow)$ be an alternative quadri-algebra and $\alpha : A \to A$ an alternative quadri-algebra endomorphism. Define $\searrow_\alpha, \nearrow_\alpha, \swarrow_\alpha, \nwarrow_\alpha : A \times A \to A$ by*

$$\begin{aligned}
x \searrow_\alpha y &= \alpha(x) \searrow \alpha(y), & x \nearrow_\alpha y &= \alpha(x) \nearrow \alpha(y),\\
x \swarrow_\alpha y &= \alpha(x) \swarrow \alpha(y), & x \nwarrow_\alpha y &= \alpha(x) \nwarrow \alpha(y),
\end{aligned}$$

*for all $x, y \in A$. Then $A_\alpha := (A, \nwarrow_\alpha, \swarrow_\alpha, \nearrow_\alpha, \searrow_\alpha, \alpha)$ is a multiplicative Hom-alternative quadri-algebra, called the Yau twist of $A$. Moreover, assume that $(A', \nwarrow', \swarrow', \nearrow', \searrow')$ is another alternative quadri-algebra and $\alpha' : A' \to A'$ is an alternative quadri-algebra morphism and $f : A \to A'$ is an alternative*

*quadri-algebra morphism satisfying* $f\alpha = \alpha' f$. *Then* $f : A_\alpha \to A'_{\alpha'}$ *is a Hom-alternative quadri-algebra endomorphism.*

*Proof.* We define the following operations $x \succ_\alpha y := x \nearrow_\alpha y + x \searrow_\alpha y$, $x \prec_\alpha y := x \nwarrow_\alpha y + x \swarrow_\alpha y$, $x \vee_\alpha y := x \searrow_\alpha y + x \swarrow_\alpha y$, $x \wedge_\alpha y := x \nearrow_\alpha y + x \nwarrow_\alpha y$ and $x *_\alpha y := x \searrow_\alpha y + x \nearrow_\alpha y + x \swarrow_\alpha y + x \nwarrow_\alpha y$, for all $x, y \in A$. It is easy to get $x \succ_\alpha y = \alpha(x) \succ \alpha(y), x \prec_\alpha y = \alpha(x) \prec \alpha(y), x \vee_\alpha y = \alpha(x) \vee \alpha(y), x \wedge_\alpha y = \alpha(x) \wedge \alpha(y)$ and $x *_\alpha y = \alpha(x) * \alpha(y)$ for all $x, y \in A$. Since $\alpha$ is a quadri-alternative algebra endomorphism, for all $x, y, z \in A$,

$$\begin{aligned}
(\!(x,y,z)\!)^{ne}_\alpha + (\!(y,x,z)\!)^{e}_\alpha &= (x \wedge_\alpha y) \nearrow_\alpha \alpha(z) - \alpha(x) \nearrow_\alpha (y \succ_\alpha z) \\
&\quad + (y \vee_\alpha x) \nearrow_\alpha \alpha(z) - \alpha(y) \searrow_\alpha (x \nearrow_\alpha z) \\
&= (\alpha^2(x) \wedge \alpha^2(y)) \nearrow \alpha^2(z) \\
&\quad - \alpha^2(x) \nearrow (\alpha^2(y) \succ \alpha^2(z)) \\
&\quad + (\alpha^2(y) \vee \alpha^2(x)) \nearrow \alpha^2(z) \\
&\quad - \alpha^2(y) \searrow (\alpha^2(x) \nearrow \alpha^2(z)) \\
&= \alpha^2(\!(x,y,z)\!)^{ne} + \alpha^2(\!(y,x,z)\!)^{e} = 0.
\end{aligned}$$

□

**Remark 5.2.** Let $(A, \nwarrow, \swarrow, \nearrow, \searrow, \alpha)$ be a Hom-alternative quadri-algebra and $\gamma : A \to A$ be a Hom-alternative quadri-algebra morphism. Define new multiplications on $A$ by

$$\begin{aligned}
x \nearrow' y &= \gamma(x) \nearrow \gamma(y), \quad x \searrow' y = \gamma(x) \searrow \gamma(y), \\
x \nwarrow' y &= \gamma(x) \nwarrow \gamma(y), \quad x \swarrow' y = \gamma(x) \swarrow \gamma(y).
\end{aligned}$$

Then, $(A', \nearrow', \searrow', \swarrow', \nwarrow', \alpha \circ \gamma)$ is a multiplicative Hom-alternative quadri-algebra.

**Definition 5.5.** Let $(A, \prec, \succ, \alpha)$ be a Hom-pre-alternative algebra. A linear map $T : V \to A$ is called an $\mathcal{O}$-operator of $(A, \prec, \succ, \alpha)$ associated to a representation $(V, L_\prec, R_\prec, L_\succ, R_\succ, \beta)$ if for all $a, b \in V$,

$$\begin{aligned}
T \circ \beta &= \alpha \circ T, \\
T(a) \succ T(b) &= T\big(L_\succ(T(a))b + R_\succ(T(b))a\big), \\
T(a) \prec T(b) &= T\big(L_\prec(T(a))b + R_\prec(T(b))a\big).
\end{aligned} \tag{5.16}$$

We have the following results.

**Proposition 5.4.** *Let* $(V, L_\prec, R_\prec, L_\succ, R_\succ, \beta)$ *be a representation of a Hom-pre-alternative algebra* $(A, \prec, \succ, \alpha)$ *and* $(A, *, \alpha)$ *be the associated Hom-alternative algebra. If* $T$ *is an* $\mathcal{O}$*-operator of* $(A, \prec, \succ, \alpha)$ *associated to* $(V, L_\prec, R_\prec, L_\succ, R_\succ, \beta)$, *then* $T$ *is an* $\mathcal{O}$*-operator of* $(A, \circ, \alpha)$ *associated to* $(V, L_\prec + L_\succ, R_\prec + R_\succ, \beta)$.

**Proposition 5.5.** *If* $(A, \prec, \succ, \alpha)$ *is a Hom-pre-alternative algebra and* $T$ *is an* $\mathcal{O}$*-operator of* $(A, \prec, \succ, \alpha)$ *associated to a representation* $(V, L_\prec, R_\prec, L_\succ,$

$R_\succ, \beta)$, *then* $(V, \searrow, \nearrow, \swarrow, \nwarrow, \beta)$ *is a Hom-alternative quadri-algebra with products defined, for all* $a, b \in V$, *by*

$$\begin{aligned} a \searrow b &= L_\succ(T(a))b,\ a \nearrow b = R_\succ(T(b))a,\\ a \swarrow b &= L_\prec(T(a))b,\ a \nwarrow b = R_\prec(T(b))a. \end{aligned} \tag{5.17}$$

*Proof.* Set $L = L_\prec + L_\succ$ and $R = R_\prec + R_\succ$. For any $a, b, c \in V$,

$$\begin{aligned} &(a * b) \searrow \beta(c) - \beta(a) \searrow (b \searrow c)\\ &\quad = (L(T(a))b + R(T(b))a) \searrow \beta(c) - \beta(a) \searrow (L_\succ(T(b))c)\\ &\quad = L_\succ(T(L(T(a))b + R(T(b))a))\beta(c) - L_\succ(\alpha(T(a)))L_\succ(T(b))c\\ &\quad = L_\succ(T(a) * T(b))\beta(c) - L_\succ(\alpha(T(a)))L_\succ(T(b))c = 0.\\ &(a \swarrow b) \nwarrow \beta(c) - \phi(a) \swarrow (b \nwarrow c + b \nearrow c)\\ &\quad = R_\prec(\alpha(T(c)))L_\prec(T(a))b - L_\prec(\alpha(T(a)))(R_\prec(T(c))b + R_\succ(T(c))b)\\ &\quad = R_\prec(\alpha(T(c)))L_\prec(T(a))b - L_\prec(\alpha(T(a)))R(T(c))b,\\ &(a \nwarrow c + a \swarrow c) \swarrow \beta(b) - \beta(a) \swarrow (c \searrow b + c \swarrow b)\\ &\quad = L_\prec(T(R_\prec(T(c))a + L_\prec(T(a))c))\beta(b)\\ &\quad - L_\prec(\alpha(T(a)))(L_\succ(T(c))b + L_\prec(T(c))b)\\ &\quad = L_\prec(T(a) \prec T(c))\beta(b) - L_\prec(\alpha(T(a)))(L(T(c))b. \end{aligned}$$

This means that

$$\begin{aligned} (\!(a, b, c)\!)^w_\beta + (\!(a, c, b)\!)^{sw}_\beta =& R_\prec(\alpha(T(c)))L_\prec(T(a))b - L_\prec(\alpha(T(a)))R(T(c))b\\ &+ L_\prec(T(a) \prec T(c))\beta(b)\\ &- L_\prec(\alpha(T(a)))L(T(c))b = 0, \end{aligned}$$

since $(V, L_\prec, R_\prec, L_\succ, R_\succ, \beta)$ is a representation of $(A, \prec, \succ, \alpha)$. Similarly, we can proof the others identities using analogous computations.

□

Rota–Baxter operators $\mathcal{R} : A \to A$ are a special case of $\mathcal{O}$-operators on a Hom-pre-alternative algebra $(A, \prec, \succ, \alpha)$ satisfying the following conditions, for all $x, y \in A$,

$$\mathcal{R} \circ \alpha = \alpha \circ \mathcal{R}, \tag{5.18}$$

$$\mathcal{R}(x) \succ \mathcal{R}(y) = \mathcal{R}(x \succ \mathcal{R}(y) + \mathcal{R}(x) \succ y), \tag{5.19}$$

$$\mathcal{R}(x) \prec \mathcal{R}(y) = \mathcal{R}(x \prec \mathcal{R}(y) + \mathcal{R}(x) \prec y). \tag{5.20}$$

Analogously to Proposition 5.5, we have the following construction.

**Corollary 5.2.** *Let* $(A, \prec, \succ, \alpha)$ *be a Hom-pre-alternative algebra and* $\mathcal{R} : A \to A$ *be a Rota–Baxter operator of weight* 0 *for* $A$. *Then* $(A, \nearrow, \searrow, \swarrow, \nwarrow, \alpha)$ *is a Hom-alternative quadri-algebra where the operations* $\nearrow, \searrow, \swarrow, \nwarrow$ *are given, for* $x,\ y \in A$, *by*

$$\begin{aligned} x \nearrow y &= x \succ \mathcal{R}(y), & x \searrow y &= \mathcal{R}(x) \succ y,\\ x \swarrow y &= \mathcal{R}(x) \prec y, & x \nwarrow y &= x \prec \mathcal{R}(y). \end{aligned}$$

**Proposition 5.6.** *Let $(A, *, \alpha)$ be a Hom-alternative algebra and $\mathcal{R}_1, \mathcal{R}_2$ be two commuting Rota–Baxter operators on $A$ such that $\mathcal{R}_1\alpha = \alpha\mathcal{R}_1$ and $\mathcal{R}_2\alpha = \alpha\mathcal{R}_2$. Then $\mathcal{R}_2$ is a Rota–Baxter operator on the Hom-pre-alternative algebra $(A, \prec, \succ, \alpha)$. where $x \prec_R y = x * \mathcal{R}_1(y)$ and $x \succ_R y = \mathcal{R}_1(x) * y$. Moreover, there exists a Hom-alternative quadri-algebra structure on the underlying vector space $(A, *, \alpha)$, with operations defined by*

$$\begin{aligned} x \searrow y &= \mathcal{R}_2(x) \succ y = \mathcal{R}_1(\mathcal{R}_2(x)) * y, \\ x \nearrow y &= x \succ \mathcal{R}_2(y) = \mathcal{R}_1(x) * \mathcal{R}_2(y), \\ x \swarrow y &= \mathcal{R}_2(x) \prec y = \mathcal{R}_2(x) * \mathcal{R}_1(y), \\ x \nwarrow y &= x \prec \mathcal{R}_2(y) = x * \mathcal{R}_1(\mathcal{R}_2(y)). \end{aligned}$$

*Proof.* For all $x, y \in A$,

$$\begin{aligned} \mathcal{R}_2(x) \succ \mathcal{R}_2(y) &= \mathcal{R}_1(\mathcal{R}_2(x)) * \mathcal{R}_2(y) = \mathcal{R}_2(\mathcal{R}_1(x)) * \mathcal{R}_2(y) \\ &= \mathcal{R}_2(\mathcal{R}_1(x) * \mathcal{R}_2(y) + \mathcal{R}_2(\mathcal{R}_1(x)) * y) \\ &= \mathcal{R}_2(\mathcal{R}_1(x) * \mathcal{R}_2(y) + \mathcal{R}_1(\mathcal{R}_2(x)) * y) \\ &= \mathcal{R}_2(x \succ \mathcal{R}_2(y) + \mathcal{R}_2(x) \succ y), \\ \mathcal{R}_2(x) \prec \mathcal{R}_2(y) &= \mathcal{R}_2(x) * \mathcal{R}_1(\mathcal{R}_2(y)) = \mathcal{R}_2(x) * \mathcal{R}_2(\mathcal{R}_1(y)) \\ &= \mathcal{R}_2(x * \mathcal{R}_2(\mathcal{R}_1(y)) + \mathcal{R}_2(x) * \mathcal{R}_1(y)) \\ = \mathcal{R}_2(x * \mathcal{R}_1(\mathcal{R}_2(y)) + \mathcal{R}_2(x) * \mathcal{R}_1(y)) & \\ &= \mathcal{R}_2(x \prec \mathcal{R}_2(y) + \mathcal{R}_2(x) \prec y). \end{aligned}$$

On the other hand, we use the construction in Corollary 5.2 with the Rota–Baxter operator $\mathcal{R}_2$ of weight 0 and the Hom-pre-alternative algebra $(A, \prec, \succ, \alpha)$ .

□

**Theorem 5.1.** *Assume the hypothesis of Corollary 5.1. Let $(V, L_\prec, R_\prec, L_\succ, R_\succ, \beta)$ be a representation and $T$ be an $\mathcal{O}$-operator of $(A, \prec, \succ, \alpha)$ associated with $(V, L_\prec, R_\prec, L_\succ, R_\succ, \beta)$. Then $(V, L_\succ - R_\prec, R_\succ - L_\prec, \beta)$ is a representation of the Hom-pre-Malcev algebra $(A, \cdot, \alpha)$, and $T$ is an $\mathcal{O}$-operator of $(A, \cdot, \alpha)$ with respect to $(V, L_\succ - R_\prec, R_\succ - L_\prec, \beta)$. Moreover, if $(V, \blacktriangleright, \blacktriangleleft, \beta)$ is the Hom-M-dendriform algebra associated to the Hom-pre-Malcev algebra $(A, \cdot, \alpha)$ on the representation $(V, L_\succ - R_\prec, R_\succ - L_\prec, \beta)$, then*

$$a \blacktriangleright b = a \nearrow b - b \swarrow a, \qquad a \blacktriangleleft b = a \searrow b - b \nwarrow a.$$

*Proof.* By Proposition 5.1, $A \ltimes V$ is a Hom-pre-alternative algebra. Consider its associated Hom-pre-Malcev algebra $(A \oplus V, \bullet, \alpha + \beta)$,

$$\begin{aligned} &(x + a) \bullet (y + b) = (x + a) \gg (y + b) - (y + b) \ll (x + a) \\ &= x \succ y + L_\succ(x)b + R_\succ(y)a - y \prec x - L_\prec(y)a - R_\prec(x)b \\ &= x \cdot y + (L_\succ - R_\prec)(x)b + (R_\succ - L_\prec)(y)a. \end{aligned}$$

By Proposition 3.1, $(V, L_\succ - R_\prec, R_\succ - L_\prec, \beta)$ is a representation of the Hom-pre-Malcev algebra $(A, \cdot, \alpha)$. In addition, $T$ is $\mathcal{O}$-operator of $(A, \cdot, \alpha)$ with respect to $(V, L_\succ - R_\prec, R_\succ - L_\prec, \beta)$:

$$T(a) \cdot T(b) = T(a) \succ T(b) - T(b) \prec T(a)$$

$$
\begin{aligned}
&= T\big(L_{\succ}(T(a))b + R_{\succ}(T(b))a\big) - T\big(L_{\prec}(T(b))a + R_{\prec}(T(a))b\big)\\
&= T\big((L_{\succ} - R_{\prec})(T(a))b + (R_{\succ} - L_{\prec})(T(b))a\big).
\end{aligned}
$$

Moreover, by (4.12) and (5.16),

$$
\begin{aligned}
a \blacktriangleright b &= (R_{\succ} - L_{\prec})(T(b))a = R_{\succ}(T(b))a - L_{\prec}(T(b))a = a \nearrow b - b \swarrow a,\\
a \blacktriangleleft b &= (L_{\succ} - R_{\prec})(T(a))b = L_{\succ}(T(a))b - R_{\prec}(T(a))b = a \searrow b - b \nwarrow a.
\end{aligned}
$$

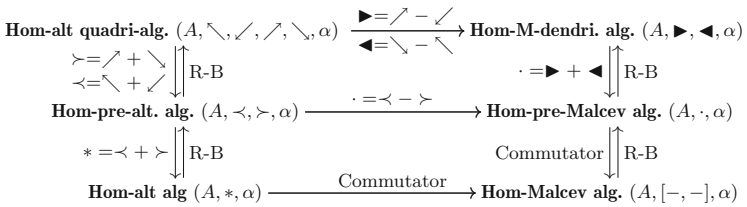
□

**Corollary 5.3.** *Assume hypothesis of Corollary 5.1. Let $\mathcal{R}$ be a Rota–Baxter operator of $(A, \prec, \succ, \alpha)$ and $(A, \blacktriangleright, \blacktriangleleft, \alpha)$ be the Hom-M-dendriform algebra associated to the Hom-pre-Malcev algebra $(A, \cdot, \alpha)$ given in Corollary 4.2. Then, the operations*

$$
x \blacktriangleright y = x \nearrow y - y \swarrow x, \qquad x \blacktriangleleft y = x \searrow y - y \nwarrow x,
$$

*define a Hom-M-dendriform structure in $A$ with respect to the twisting map $\alpha$.*

Summarising the above study in this section, we have the following commutative diagram:

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